PART B

Unit - V

Special Functions

5.1 Introduction

The Laplace equation $\nabla^2 u = 0$ in various orthogonal curvilinear coordinate systems is of great importance in many physical and engineering problems. The solution of Laplace's equation in cylindrical system and spherical system leads to two important ordinary differential equations namely the Bessel differential equation and Legendre differential equation respectively. The series solution of the Bessel's differential equation is a special function known as the Bessel function. The special polynomial function that occurs in the process of solving in series the Legendre's differential equation is known as Legendre polynomial.

The reader is familiar with series solution method for solving homogeneous ordinary differential equation. [*Refer Volume - II*]

The Bessel function has various applications in solving boundary value problems with axial symmetry and the Legendre polynomial has various applications in solving boundary value problems with spherical symmetry.

5.2 Solution of Laplace Equation in Cylindrical System Leading to Bessel Differential Equation

The coordinates (ρ, ϕ, z) are called the cylindrical coordinates and the relationship with the cartesian coordinates (x, y, z) is given by $x = \rho \cos\phi$, $y = \rho \sin\phi$, $z = z$.

The Laplace equation $\nabla^2 f = 0$ in the cylindrical system is given by

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 \quad \text{[Refer Vol-I, Page -317]} \qquad \ldots (1)$$

We shall solve this by the method of separation of variables (*Product method*)

Let $f = f_1 f_2 f_3$ be the solution of (1), where $f_1 = f_1(\rho)$, $f_2 = f_2(\phi)$, $f_3 = f_3(z)$.

Substituting this in (1) we have,

$$\frac{\partial^2 (f_1 f_2 f_3)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial (f_1 f_2 f_3)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 (f_1 f_2 f_3)}{\partial \phi^2} + \frac{\partial^2 (f_1 f_2 f_3)}{\partial z^2} = 0$$

That is, $f_2 f_3 \dfrac{d^2 f_1}{d\rho^2} + \dfrac{f_2 f_3}{\rho} \dfrac{d f_1}{d\rho} + \dfrac{f_1 f_3}{\rho^2} \dfrac{\partial^2 f_2}{d\phi^2} + f_1 f_2 \dfrac{d^2 f_3}{dz^2} = 0$

Dividing by $f_1 \; f_2 \; f_3$ we have,

$$\frac{1}{f_1} \frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho f_1} \frac{df_1}{d\rho} + \frac{1}{\rho^2 f_2} \frac{d^2 f_2}{d\phi^2} + \frac{1}{f_3} \frac{d^2 f_3}{dz^2} = 0$$

i.e.,
$$\frac{1}{f_1} \frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho f_1} \frac{df_1}{d\rho} + \frac{1}{\rho^2 f_2} \frac{d^2 f_2}{d\phi^2} = -\frac{1}{f_3} \frac{d^2 f_3}{dz^2} \qquad \ldots (2)$$

The LHS is a function of $\rho, \; \phi$ and RHS is a function of z. Therefore they must be equal to a constant.

Let us set $\dfrac{1}{f_3} \dfrac{d^2 f_3}{dz^2} = 1$, so that (2) becomes

$$\frac{1}{f_1} \frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho f_1} \frac{df_1}{d\rho} + \frac{1}{\rho^2 f_2} \frac{d^2 f_2}{d\phi^2} = -1$$

Now multiplying by ρ^2 we get,

$$\frac{\rho^2}{f_1} \frac{d^2 f_1}{d\rho^2} + \frac{\rho}{f_1} \frac{df_1}{d\rho} + \frac{1}{f_2} \frac{d^2 f_2}{d\phi^2} = -\rho^2$$

or
$$\frac{\rho^2}{f_1} \frac{d^2 f_1}{d\rho^2} + \frac{\rho}{f_1} \frac{df_1}{d\rho} + \rho^2 = \frac{-1}{f_2} \frac{d^2 f_2}{d\phi^2} \qquad \ldots (3)$$

Again, LHS is a function of ρ and RHS is a function of ϕ. Therefore they must be equal to a constant.

Now setting $\dfrac{-1}{f_2} \dfrac{d^2 f_2}{d\phi^2} = n^2$, (3) becomes

$$\frac{\rho^2}{f_1} \frac{d^2 f_1}{d\rho^2} + \frac{\rho}{f_1} \frac{df_1}{d\rho} + \rho^2 = n^2$$

or
$$\frac{\rho^2}{f_1} \frac{d^2 f_1}{d\rho^2} + \frac{\rho}{f_1} \frac{df_1}{d\rho} + (\rho^2 - n^2) = 0$$

i.e.,
$$\rho^2 \frac{d^2 f_1}{d\rho^2} + \rho \frac{df_1}{d\rho} + (\rho^2 - n^2) f_1 = 0$$

This equation can be written in the form

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$$

This is the ***Bessel's differential equation*** of order n in the standard form originating from the Laplace equation in the cylindrical system.

5.3 Solution of Laplace Equation in Spherical System Leading to Legendre Differential Equation

The coordinates (r, θ, ϕ) are called the spherical polar coordinates and the relationship with the cartesian coordinates (x, y, z) is given by $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$.

The Laplace equation $\nabla^2 f = 0$ in the spherical system is given by

$$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} = 0 \qquad \ldots(1)$$

[*Refer Vol-1, Page - 317*]

We shall solve this by the method of separation of variables (*Product method*).

Let $f = f_1 f_2 f_3$ be the solution of (1) where $f_1 = f_1(r), f_2 = f_2(\theta), f_3 = f_3(\phi)$.

Substituting this in (1) we have,

$$\frac{\partial^2 (f_1 f_2 f_3)}{\partial r^2} + \frac{2}{r}\frac{\partial (f_1 f_2 f_3)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial (f_1 f_2 f_3)}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 (f_1 f_2 f_3)}{\partial \phi^2} = 0$$

i.e.,

$$f_2 f_3 \frac{d^2 f_1}{dr^2} + \frac{2 f_2 f_3}{r}\frac{df_1}{dr} + \frac{f_1 f_3}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{df_2}{d\theta}\right) + \frac{f_1 f_2}{r^2 \sin^2\theta}\frac{d^2 f_3}{d\phi^2} = 0$$

Dividing by $f_1 f_2 f_3$ we have,

$$\frac{1}{f_1}\frac{d^2 f_1}{dr^2} + \frac{2}{r f_1}\frac{df_1}{dr} + \frac{1}{r^2 \sin\theta f_2}\left(\sin\theta \frac{d^2 f_2}{d\theta^2} + \frac{df_2}{d\theta}\cos\theta\right) + \frac{1}{r^2 \sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2} = 0$$

i.e., $$\left(\frac{1}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2}{rf_1}\frac{df_1}{dr}\right)+\left(\frac{1}{r^2 f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{r^2 f_2}\frac{df_2}{d\theta}\right)+\frac{1}{r^2\sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2}=0$$

Multiplying by r^2 we have,

$$\left(\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}\right)+\left(\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right)+\frac{1}{\sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2}=0$$

i.e., $$\sin^2\theta\left(\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}\right)+\sin^2\theta\left(\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right)=-\frac{1}{f_3}\frac{d^2 f_3}{d\phi^2} \quad \ldots(2)$$

Since LHS is a function of r, θ and RHS is a function of ϕ, they must be equal to a constant.

Setting $\frac{1}{f_3}\frac{d^2 f_3}{d\phi^2}=0$, equation (2) on dividing by $\sin^2\theta$ will give us

$$\left(\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}\right)+\left(\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right)=0$$

i.e., $$\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}=-\left(\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}\right) \quad \ldots(3)$$

Again LHS is a function of θ and RHS is a function of r, with the result they must be equal to a constant.

Now setting $\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}=n(n+1)$ we obtain

$$\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}=-n(n+1)$$

or $$\frac{d^2 f_2}{d\theta^2}+\cot\theta\frac{df_2}{d\theta}+n(n+1)f_2=0 \quad \ldots(4)$$

Now by taking $x=\cos\theta$, we shall convert the differential equation given by (4) in terms of f_2 and x as follows.

$$\frac{df_2}{d\theta}=\frac{df_2}{dx}\cdot\frac{dx}{d\theta}=\frac{df_2}{dx}(-\sin\theta)=-\sin\theta\frac{df_2}{dx} \quad \ldots(5)$$

$$\frac{d^2 f_2}{d\theta^2} = \frac{d}{d\theta}\left(\frac{df_2}{d\theta}\right) = \frac{d}{d\theta}\left(-\sin\theta\,\frac{df_2}{dx}\right)$$

$$= -\sin\theta\,\frac{d}{d\theta}\left(\frac{df_2}{dx}\right) + \frac{df_2}{dx}\,(-\cos\theta)$$

$$= -\sin\theta\cdot\frac{d}{dx}\left(\frac{df_2}{dx}\right)\frac{dx}{d\theta} - \frac{df_2}{dx}\cos\theta$$

$$= -\sin\theta\,\frac{d^2 f_2}{dx^2}(-\sin\theta) - \frac{df_2}{dx}\cos\theta$$

i.e., $$\frac{d^2 f_2}{d\theta^2} = \sin^2\theta\,\frac{d^2 f_2}{dx^2} - \cos\theta\,\frac{df_2}{dx} \qquad \ldots(6)$$

Hence (4) as a consequence of (5) and (6) becomes,

$$\sin^2\theta\,\frac{d^2 f_2}{dx^2} - \cos\theta\,\frac{df_2}{dx} + \cot\theta\left(-\sin\theta\,\frac{df_2}{dx}\right) + n(n+1)f_2 = 0$$

i.e., $$\sin^2\theta\,\frac{d^2 f_2}{dx^2} - \cos\theta\,\frac{df_2}{dx} - \cos\theta\,\frac{df_2}{dx} + n(n+1)f_2 = 0$$

i.e., $$(1-\cos^2\theta)\,\frac{d^2 f_2}{dx^2} - 2\cos\theta\,\frac{df_2}{dx} + n(n+1)f_2 = 0$$

Since $x = \cos\theta$, the equation becomes

$$(1 - x^2)\frac{d^2 f_2}{dx^2} - 2x\,\frac{df_2}{dx} + n(n+1)f_2 = 0$$

This equation can be written in the following standard form.

$$(1-x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

This is the ***Legendre's differential equation*** in the standard form originating from the Laplace equation in the spherical system.

5.4 Series Solution of Bessel's Differential Equation Leading to Bessel Functions

Preamble : *Refer Volume-II, Page - 151 for series solution of ODE*

The Bessel differential equation of order n is in the form,

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0 \qquad \ldots (1)$$

where n is a non negative real constant. (*parameter*)
We employ Frobenius method to solve this equation as we have,
coefficient of $y'' = x^2 = P_0(x)$ (say) and $P_0(x) = 0$ at $x = 0$.

We assume the series solution of (1) in the form

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \qquad \ldots (2)$$

$$\therefore \quad \frac{dy}{dx} = \sum_{0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$\frac{d^2 y}{dx^2} = \sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

Now (1) becomes,

$$\sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{0}^{\infty} a_r (k+r) x^{k+r} + \sum_{0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{0}^{\infty} a_r x^{k+r} = 0$$

Collecting the first, second and fourth terms together we have,

$$\sum_{0}^{\infty} a_r x^{k+r} \left[(k+r)(k+r-1) + (k+r) - n^2 \right] + \sum_{0}^{\infty} a_r x^{k+r+2} = 0$$

i.e., $$\sum_{0}^{\infty} a_r x^{k+r} \left[(k+r) \{\overline{k+r-1} + 1\} - n^2 \right] + \sum_{0}^{\infty} a_r x^{k+r+2} = 0$$

i.e., $$\sum_{0}^{\infty} a_r x^{k+r} \left[(k+r)^2 - n^2 \right] + \sum_{0}^{\infty} a_r x^{k+r+2} = 0$$

We shall equate the coefficient of the lowest degree term in x, that is x^k to zero.

i.e., $\quad a_0 (k^2 - n^2) = 0.$

Setting $a_0 \neq 0$ we have $k^2 - n^2 = 0$ and hence $k = \pm n$

Also we need to independently equate the coefficient of x^{k+1} to zero.

i.e., $a_1 [(k+1)^2 - n^2] = 0.$

This implies $a_1 = 0$ since $(k+1)^2 - n^2 = 0$ would mean $(k+1)^2 = n^2$ or $(k+1) = \pm n$ which cannot be accepted as we have already $k = \pm n$.

Next, we shall equate the coefficient of x^{k+r} $(r \geq 2)$ to zero.

i.e., $a_r [(k+r)^2 - n^2] + a_{r-2} = 0$

or $$a_r = \frac{-a_{r-2}}{[(k+r)^2 - n^2]} \quad (r \geq 2) \qquad \ldots (3)$$

When $k = +n$, (3) becomes,

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} = \frac{-a_{r-2}}{2nr + r^2}$$

Putting $r = 2, 3, 4, \ldots$ we obtain,

$$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)} \; ; \quad a_3 = \frac{-a_1}{6n+9} = 0 \text{ since } a_1 = 0.$$

Similarly $a_5, a_7, \ldots$ are all equal to zero.

i.e., $a_1 = 0 = a_3 = a_5 = a_7 = \ldots.$

Next, $$a_4 = \frac{-a_2}{8n+16} = \frac{-a_2}{8(n+2)} = \frac{a_0}{32(n+1)(n+2)}$$ and so on.

We substitute these values in the expanded form of (2) :

$$y = x^k (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots)$$

Also let the solution for $k = +n$ be denoted by y_1.

$$\therefore \quad y_1 = x^n \left[a_0 - \frac{a_0}{4(n+1)} x^2 + \frac{a_0}{32(n+1)(n+2)} x^4 - \ldots \right]$$

i.e., $$y_1 = a_0 x^n \left[1 - \frac{x^2}{2^2 (n+1)} + \frac{x^4}{2^5 (n+1)(n+2)} - \ldots \right] \qquad \ldots (4)$$

Since we also have $k = -n$, let thesolutionfor $k = -n$ be denoted by y_2.

Replacing n by $-n$ in (4) we have,

$$y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{2^2 (-n+1)} + \frac{x^4}{2^5 (-n+1)(-n+2)} - \ldots \right] \qquad \ldots (5)$$

The complete (*general*) solution of (1) is given by

$y = A y_1 + B y_2$, where A, B are arbitrary constants.

We shall now standardize the solution as in (4) by choosing

$$a_0 = \frac{1}{2^n \Gamma(n+1)} \quad \text{and the same be denoted by } Y_1.$$

$$Y_1 = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\cdot 2} - \cdots \right]$$

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma(n+1)\cdot 2} - \cdots \right]$$

We have a property of gamma functions,

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$\therefore$ $\Gamma(n+2) = (n+1)\Gamma(n+1)$ and

$$\Gamma(n+3) = (n+2)\Gamma(n+2) = (n+2)(n+1)\Gamma(n+1)$$

As a consequence of these results we now have,

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(n+3)\cdot 2} - \cdots \right]$$

This can further be put in the form

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\Gamma(n+1)\cdot 0!} \left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2)\cdot 1!} \left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3)\cdot 2!} \left(\frac{x}{2}\right)^4 + \cdots \right]$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\cdot r!} \left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!}$$

This function is called the ***Bessel function of the first kind of order n denoted by*** $J_n(x)$.

Thus $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!} \qquad \ldots (6)$$

Further, the solution for $k = -n$, (*in respect of* y_2) be denoted by $J_{-n}(x)$.

Hence the general solution of the Bessel's equation is given by

$$y = aJ_n(x) + bJ_{-n}(x)$$

where a and b are arbitrary constants and n is not an integer.

(*See the Remark in the next article regarding* n *not to be an integer.*)

5.41 Equation reducible to the form of Bessel's equation

Consider the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda^2 x^2 - n^2) y = 0 \qquad \ldots (1)$$

We shall show that this equation is reducible to the form of Bessel equation.

Putting $t = \lambda x$ we have,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \lambda \frac{dy}{dt}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\lambda \frac{dy}{dt}\right) = \frac{d}{dt}\left(\lambda \frac{dy}{dt}\right)\frac{dt}{dx} = \lambda^2 \frac{d^2 y}{dt^2}$$

Substituting these results along with $x = t/\lambda$ in (1) we obtain,

$$\frac{t^2}{\lambda^2} \cdot \lambda^2 \frac{d^2 y}{dt^2} + \frac{t}{\lambda} \lambda \frac{dy}{dt} + (t^2 - n^2) y = 0$$

i.e., $$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - n^2) y = 0$$

This is in the form of Bessel differential equation whose solution is given by $y = aJ_n(t) + bJ_{-n}(t)$.

Thus $y = aJ_n(\lambda x) + bJ_{-n}(\lambda x)$ is the solution of equation (1).

5.42 Properties of Bessel functions

Property 1. $J_{-n}(x) = (-1)^n J_n(x)$ *where* n *is a positive integer.*

Proof : By the definition of Bessel function we have,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \quad \ldots(1)$$

$$\therefore \quad J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)\, r!} \quad \ldots(2)$$

In (2) $\Gamma(-n+r+1) = \Gamma[r-(n-1)]$ is of the form $\Gamma(-k)$ for $r = 0, 1, 2, \cdots (n-2)$ and $\Gamma(0)$ for $r = (n-1)$.

Noting that $\Gamma(-k) \to \infty$ or $\dfrac{1}{\Gamma(-k)} \to 0$, k being a positive integer we can say that $\dfrac{1}{\Gamma[r-(n-1]} \to 0$ for $r = 0, 1, 2 \cdots (n-1)$

$$\text{Hence, } J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)\, r!} \quad \ldots(3)$$

Let $r - n = s$ or $r = s + n$ so that we have when $r = n,\ s = 0$.

Now (3) assumes the form

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{-n+2s+2n} \frac{1}{\Gamma(s+1)(s+n)!}$$

$$= \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{n+2s} \frac{1}{\Gamma(s+1)(s+n)!}$$

Using the properties of gamma function we can write $\Gamma(s+1) = s!$ and $(s+n)! = \Gamma(s+n+1)$

$$\therefore \quad J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{n+2s} \frac{1}{s!\,\Gamma(s+n+1)}$$

$$= (-1)^n \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \frac{1}{\Gamma(n+s+1)\, s!}$$

Comparing with (1) we observe that the summation in the RHS is $J_n(x)$.

Thus we have proved that $J_{-n}(x) = (-1)^n J_n(x)$, n being a positive integer.

Remark : *From this property we can easily conclude that $J_{-n}(x)$ and $J_n(x)$ are not linearly independent when n is an integer. Hence the general solution of the Bessel's differential equation is $y = a J_n(x) + b J_{-n}(x)$ when **n is not an integer.** Equivalently, we can say that $J_n(x)$ and $J_{-n}(x)$ are linearly independent solutions of the Bessel's equation when n is not an integer.*

Property 2. $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$ *where n is a positive integer.*

Proof : We have $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$\therefore \quad J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \left(-\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$= \sum_{r=0}^{\infty} (-1)^r (-1)^{n+2r} \frac{x^{n+2r}}{2^{n+2r}} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} \left\{(-1)^3\right\}^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

Thus $J_n(-x) = (-1)^n J_n(x)$

Since $(-1)^n J_n(x) = J_{-n}(x)$ we have,

$$J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$$

5.43 Recurrence relations / Recurrence formulae

We derive recurrence relations relating to Bessel function of different orders from the basic definition of $J_n(x)$.

1. $2n J_n(x) = x[J_{n+1}(x) + J_{n-1}(x)]$

Proof : We have by the definition

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$\therefore \quad 2n J_n(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2n}{\Gamma(n+r+1)\, r!}$$

We shall write $2n = 2(n+r) - 2r$ and split the summation into two terms.

i.e.,
$$2nJ_n(x) = \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2(n+r)}{\Gamma(n+r+1)\,r!} - \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2r}{\Gamma(n+r+1)\,r!}$$

$$= \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2(n+r)}{(n+r)\,\Gamma(n+r)\,r!} - \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2r}{\Gamma(n+r+1)\,r\cdot(r-1)!}$$

$$= \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{2}{\Gamma(n+r)\,r!} - \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{2}{\Gamma(n+r+1)(r-1)!}$$

$$= x\sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\,r!} - x\sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)(r-1)!}$$

Putting $r-1 = s$ or $r = s+1$ in the second term of the RHS we obtain,

$$2nJ_n(x) = x\sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\,r!} - x\sum_{s=0}^{\infty} (-1)^{s+1} \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\,s!}$$

$$= x\sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\,r!} + x\sum_0^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\,s!}$$

Thus $2nJ_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$

- -

2. $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

Differentiating w.r.t. x, we have,

$$J_n'(x) = \sum_{0}^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2} \frac{1}{\Gamma(n+r+1)\, r!}$$

We shall write $n+2r = (n+r)+r$ and split the summation into two terms.

$$2J_n'(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{(n+r)}{(n+r)\,\Gamma(n+r)\, r!} + \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{r}{\Gamma(n+r+1)\, r\cdot(r-1)!}$$

$$2J_n'(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n+r)\, r!} + \sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)\,(r-1)!}$$

Putting $(r-1) = s$ in the second term of the RHS we have,

$$2J_n'(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} - \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

Thus $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

---

3. $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$\therefore \quad x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r x^{2n+2r} \frac{1}{2^{n+2r}\,\Gamma(n+r+1)\, r!}$$

Now, $$\frac{d}{dx}[x^n J_n(x)] = \sum_{0}^{\infty} (-1)^r (2n+2r)\, x^{2n+2r-1} \frac{1}{2^{n+2r}\,(n+r)\,\Gamma(n+r)\, r!}$$

$$= \sum_{0}^{\infty} (-1)^r 2(n+r)\, x^{2n+2r-1} \frac{1}{2^{n+2r}(n+r)\,\Gamma(n+r)\, r!}$$

$$= \sum_{0}^{\infty} (-1)^r \cdot x^n x^{n+2r-1} \cdot \frac{1}{2^{n+2r-1}\,\Gamma(n+r)\, r!}$$

$$= x^n \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!}$$

$$= x^n J_{n-1}(x)$$

Thus $$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

---

4. $$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

Proof : $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$\therefore \quad x^{-n} J_n(x) = \sum_{0}^{\infty} (-1)^r x^{2r} \frac{1}{2^{n+2r}\,\Gamma(n+r+1)\, r!}$$

Now, $$\frac{d}{dx}[x^{-n} J_n(x)] = \sum_{0}^{\infty} (-1)^r 2r\, x^{2r-1} \cdot \frac{1}{2^{n+2r}\,\Gamma(n+r+1)\, r \cdot (r-1)!}$$

$$= \sum_{r=1}^{\infty} (-1)^r x^{-n} x^{n+2r-1} \frac{1}{2^{n+2r-1}\,\Gamma(n+r+1)\,(r-1)!}$$

Putting $r-1 = s$ or $r = s+1$ we have

$$\frac{d}{dx}[x^{-n} J_n(x)] = \sum_{s=0}^{\infty} (-1)^{s+1} x^{-n} x^{n+2s+1} \frac{1}{2^{n+2s+1}\,\Gamma(n+1+s+1)\, s!}$$

$$= -x^{-n} \sum_{0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

Thus $$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

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5. $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

Proof : $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!}$

$$\therefore \quad J_n'(x) = \sum_0^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2} \frac{1}{\Gamma(n+r+1)r!}$$

Hence $xJ_n'(x) = \sum_0^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!}$

We shall write $n+2r = 2(n+r) - n$ and split the summation into two terms.

$$xJ_n'(x) = \sum_0^{\infty} (-1)^r 2(n+r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{(n+r)\Gamma(n+r)r!}$$
$$+ \sum_0^{\infty} (-1)^r (-n) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!}$$

$$= \sum_0^{\infty} (-1)^r \cdot 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r)r!}$$
$$- n\sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!}$$

$$= x\sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)r!} - nJ_n(x)$$

Thus $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

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6. $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

Proof : $J_n(x) = \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!}$

$$\therefore \quad J_n'(x) = \sum_0^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2} \frac{1}{\Gamma(n+r+1)r!}$$

Hence $xJ_n'(x) = \sum_0^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)r!}$

$$x J_n'(x) = n \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$
$$+ \sum_{0}^{\infty} (-1)^r\, 2r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r \cdot (r-1)!}$$

$$= n\, J_n(x) + \sum_{r=1}^{\infty} (-1)^r \cdot 2 \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)(r-1)!}$$

Putting $r - 1 = s$ or $r = s + 1$ in the second term we have,

$$x J_n'(x) = n J_n(x) + \sum_{s=0}^{\infty} (-1)^{s+1} \cdot 2 \left(\frac{x}{2}\right)^{n+2s+2} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$= n J_n(x) - \sum_{s=0}^{\infty} (-1)^s \cdot 2 \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2s+1} \frac{1}{\Gamma(n+1+s+1)\, s!}.$$

$$= n J_n(x) - x \sum_{0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

Thus $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

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WORKED PROBLEMS

1. *Prove that*

(a) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\, \sin x$ *(b)* $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\, \cos x$

>> By the definition,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \qquad \ldots (1)$$

Putting $n = 1/2$ in (1) we have,

$$J_{1/2}(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{1/2+2r} \frac{1}{\Gamma(r+3/2)\, r!}$$
$$= \sqrt{\frac{x}{2}} \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

On expanding we have,

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(5/2)\,1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(7/2)\,2!} - \cdots\right] \quad \ldots (2)$$

We know that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(n) = (n-1)\Gamma(n-1)$

Putting $n = 3/2,\ 5/2,\ 7/2\ \ldots$ we get the following values.

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}\ ;\ \Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma(7/2) = \frac{5}{2}\Gamma(5/2) = \frac{15\sqrt{\pi}}{8}$$

Substituting these values in the RHS of (2) we have,

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{8}{15\sqrt{\pi}\cdot 2} - \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\left[2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\cdot\frac{2}{x}\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right]$$

(We have taken $2/x$ as a common factor keeping in view of the desired result)

ie., $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

Thus $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

Also by putting $n = -1/2$ in (1) we have,

$$J_{-1/2}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-1/2+2r}\frac{1}{\Gamma(r+1/2)\,r!}$$

$$= \left(\frac{x}{2}\right)^{-1/2}\sum_{0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{1}{\Gamma(r+1/2)\,r!}$$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma(1/2)} - \left(\frac{x}{2}\right)^2\frac{1}{\Gamma(3/2)\,1!} + \left(\frac{x}{2}\right)^4\frac{1}{\Gamma(5/2)\,2!} - \cdots\right]$$

Using the computed values of $\Gamma(3/2)$, $\Gamma(5/2)$ along with the value of $\Gamma(1/2)$ we have,

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{2}{\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{4}{3\sqrt{\pi}\cdot 2} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

Thus $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$

---

2. *Obtain expressions for* $J_{1/2}(x)$ *and* $J_{-1/2}(x)$. *Then use a suitable recurrence relation to deduce expressions for* $J_{3/2}(x)$ *and* $J_{-3/2}(x)$

>> We have already obtained the following results.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \ ; \ J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Let us consider the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \qquad \ldots(1)$$

Putting $n = 1/2$ in this relation we have,

$$J_{-1/2}(x) + J_{3/2}(x) = \frac{1}{x}J_{1/2}(x)$$

$$\therefore \quad J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

Thus $J_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[\dfrac{\sin x - x\cos x}{x}\right]$

Also by putting $n = -1/2$ in (1) we have,

$$J_{-3/2}(x) + J_{1/2}(x) = \frac{-1}{x}J_{-1/2}(x)$$

$$\therefore \quad J_{-3/2}(x) = -\left[J_{1/2}(x) + \frac{1}{x}J_{-1/2}(x)\right]$$

$$J_{-3/2}(x) = -\left[\sqrt{\frac{2}{\pi x}}\sin x + \frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x\right]$$

Thus $\quad J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{x\sin x + \cos x}{x}\right]$

---

3. *Show that* $J_{3/2}(x)\sin x - J_{-3/2}(x)\cos x = \sqrt{2/\pi x^3}$

>> **Note** : *Assuming the expression for* $J_{1/2}(x)$ *and* $J_{-1/2}(x)$ *we have to first establish expressions for* $J_{3/2}(x)$ *and* $J_{-3/2}(x)$ *as in Problem-2.*

Now $J_{3/2}(x)\sin x - J_{-3/2}(x)\cos x$

$$= \sqrt{\frac{2}{\pi x}}\cdot\frac{1}{x}\left[\sin^2 x - x\sin x\cos x + x\sin x\cos x + \cos^2 x\right]$$

$$= \sqrt{\frac{2}{\pi x}}\cdot\frac{1}{x}\cdot 1 = \frac{2}{\pi x^3}$$

---

4. *Prove the following results.*

(*a*) $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$

(*b*) $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right]$

>> [*Note of Problem-3 continue to hold good for this problem also*]

We consider the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

Putting $n = 3/2$ in this relation we have,

$$J_{1/2}(x) + J_{5/2}(x) = \frac{3}{x}J_{3/2}(x)$$

$$\therefore \quad J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$$

i.e., $$J_{5/2}(x) = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x - x\cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x - 3x\cos x - x^2\sin x}{x^2}\right]$$

Thus $$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{(3-x^2)\sin x}{x^2} - \frac{3\cos x}{x}\right]$$

Also by putting $n = -3/2$ in the recurrence relation we have,

$$J_{-5/2}(x) + J_{-1/2}(x) = -\frac{3}{x}J_{-3/2}(x)$$

$$\therefore \quad J_{-5/2}(x) = \frac{-3}{x}J_{-3/2}(x) - J_{-1/2}(x)$$

$$J_{-5/2}(x) = \frac{-3}{x}\cdot - \sqrt{\frac{2}{\pi x}}\left[\frac{x\sin x + \cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3x\sin x + 3\cos x - x^2\cos x}{x^2}\right]$$

Thus $$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x} + \frac{3-x^2}{x^2}\cos x\right]$$

---

5. *Starting from the expressions of $J_{1/2}(x)$ and $J_{-1/2}(x)$ in the standard form prove the following results.*

(a) $$J'_{1/2}(x)\,J_{-1/2}(x) - J'_{-1/2}(x)\,J_{1/2}(x) = \frac{2}{\pi x}$$

(b) $$\int_0^{\pi/2}\sqrt{x}\;J_{1/2}(2x)\,dx = \frac{1}{\sqrt{\pi}}$$

>> **(a)** We have results,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Differentiating these results w.r.t x we obtain,

$$J'_{1/2}(x) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{\sqrt{x}}\cos x + \sin x \cdot -\frac{1}{2}x^{-3/2}\right]$$

i.e., $$J'_{1/2}(x) = \sqrt{\frac{2}{\pi}}\left[\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x\sqrt{x}}\right]$$

$$J_{1/2}'(x) = +\sqrt{\frac{2}{\pi x}}\left(\cos x - \frac{\sin x}{2x}\right)$$

Also $$J'_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{\sqrt{x}}(-\sin x) + \cos x \cdot \frac{-1}{2}x^{-3/2}\right]$$

i.e., $$J'_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\left[-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{2x\sqrt{x}}\right]$$

$$J_{-1/2}'(x) = -\sqrt{\frac{2}{\pi x}}\left[\sin x + \frac{\cos x}{2x}\right]$$

Consider $J'_{1/2}(x)\, J_{-1/2}(x) - J'_{-1/2}(x)\, J_{1/2}(x)$

$$= \sqrt{\frac{2}{\pi x}}\left[\cos x - \frac{\sin x}{2x}\right] \cdot \sqrt{\frac{2}{\pi x}}\cos x + \sqrt{\frac{2}{\pi x}}\left[\sin x + \frac{\cos x}{2x}\right] \cdot \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \frac{2}{\pi x}(\cos^2 x + \sin^2 x) = \frac{2}{\pi x} \quad \textit{(other terms cancel out)}$$

This proves the required result.

(b) We also have, $J_{1/2}(2x) = \sqrt{\frac{2}{\pi(2x)}}\sin 2x = \frac{1}{\sqrt{\pi x}}\sin 2x$

$$\therefore \quad \sqrt{x}\, J_{1/2}(2x) = \sqrt{x} \cdot \frac{1}{\sqrt{\pi x}}\sin 2x = \frac{\sin 2x}{\sqrt{\pi}}$$

Now $$\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(2x)\, dx = \int_0^{\pi/2} \frac{\sin 2x}{\sqrt{\pi}}\, dx$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{-\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{-1}{2\sqrt{\pi}}(\cos\pi - \cos 0) = \frac{-1}{2\sqrt{\pi}}(-2) = \frac{1}{\sqrt{\pi}}$$

Thus $$\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(2x)\, dx = \frac{1}{\sqrt{\pi}}$$

---

6. *Starting from the series expression for* $J_n(x)$ *prove that*

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x).$$ *Hence deduce that*

$$J_4(x) = \frac{8}{x}\left(\frac{6}{x^2} - 1\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

\>> We have to derive the recurrence relation (1) as in the article 5.43.

That is $J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$

or $$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \qquad \dots (1)$$

Putting $n = 3$ we obtain,

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x) \qquad \dots (2)$$

We have to put $n = 1, 2$ in (1) to obtain $J_2(x)$ and $J_3(x)$ respectively.

$\therefore$ $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$ and

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

i.e., $$J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

or $$J_3(x) = \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

Using these results in the RHS of (2) we obtain

$$J_4(x) = \frac{6}{x}\left[\frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$J_4(x) = J_1(x)\left[\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right] + J_0(x)\left[1 - \frac{24}{x^2}\right]$$

Thus $$J_4(x) = \frac{8}{x}\left(\frac{6}{x^2} - 1\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

---

7. *Starting from the definition of* $J_n(x)$ *prove that*

(a) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

(b) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Hence deduce that

(c) $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

(d) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

Further deduce that

(e) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

(f) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

>> Proving *(a)* and *(b)* is obtaining the relations (3) and (4) as in the article 5.43.

Applying product rule in the LHS of (a) and (b) we obtain

$$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

and $$x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

Dividing these two equations respectively by x^n and x^{-n} we have,

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \qquad \ldots (1)$$

and $$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x) \qquad \ldots (2)$$

Multiplying both these equations by x, we get

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x)$$

and $$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

This proves the second pair of the desired relations.

Further adding and subtracting (1) and (2) we obtain

$$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

and $$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

or $\quad J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

and $\quad J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

This proves the third pair of the desired relations.

---

8. *Show that*

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right]$$

>> Consider LHS

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2J_n(x)J_n'(x) + 2J_{n+1}(x)J'_{n+1}(x) \qquad \ldots(1)$$

We have recurrence relations

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \qquad \ldots(2)$$

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x) \quad \text{or}$$

$$xJ'_{n+1}(x) = xJ_n(x) - (n+1)J_{n+1}(x) \qquad \ldots(3)$$

Let us consider the RHS of (1) and use (2) for $J_n'(x)$ and (3) for $J_{n+1}'(x)$

Hence $\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2J_n\left[\frac{n}{x}J_n - J_{n+1}\right] + 2J_{n+1}\left[J_n - \frac{n+1}{x}J_{n+1}\right]$

$$= \frac{2}{x}nJ_n^2 - 2J_nJ_{n+1} + 2J_{n+1}J_n - 2\frac{n+1}{x}J_{n+1}^2$$

$$= \frac{2}{x}nJ_n^2 - 2\frac{(n+1)}{x}J_{n+1}^2$$

Thus we have proved that,

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right]$$

---

9. *Show that* $\frac{d}{dx}\left[xJ_nJ_{n+1}\right] = x\left[J_n^2 - J_{n+1}^2\right]$

>> $\frac{d}{dx}[xJ_nJ_{n+1}] = x[J_nJ_{n+1}' + J_{n+1}J_n'] + J_nJ_{n+1} \qquad \ldots(1)$

by applying the product rule

We have recurrence relations

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \qquad \ldots (2)$$

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x) \qquad \ldots (3)$$

Replacing n by $(n+1)$, (3) becomes,

$$x J_{n+1}'(x) = x J_n(x) - (n+1) J_{n+1}(x) \qquad \ldots (4)$$

Substituting (4) and (2) in the RHS of (1) we get,

$$J_n [x J_n - (n+1) J_{n+1}] + J_{n+1} [n J_n - x J_{n+1}] + J_n J_{n+1}$$

$$= x J_n^2 - n J_n J_{n+1} - J_n J_{n+1} + n J_n J_{n+1} - x J_{n+1}^2 + J_n J_{n+1}$$

$$= x [J_n^2 - J_{n+1}^2]$$

Thus we have proved that

$$\frac{d}{dx}\left[x J_n J_{n+1}\right] = x\left[J_n^2 - J_{n+1}^2\right]$$

---

10. *Prove that* $J_0''(x) = \frac{1}{2}\left[J_2(x) - J_0(x)\right]$

>> We have the recurrence relation

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \qquad \ldots (1)$$

Putting $n = 0$, $J_0'(x) = \frac{1}{2}\left[J_{-1}(x) - J_1(x)\right]$

$$= \frac{1}{2}\left[-J_1(x) - J_1(x)\right] = -J_1(x)$$

i.e., $J_0'(x) = -J_1(x)$ and differentiating this w.r.t. x we get,

$$J_0''(x) = -J_1'(x)$$

Also from (1) when $n = 1$, $J_1'(x) = \frac{1}{2}\left[J_0(x) - J_2(x)\right]$

Hence $J_0''(x) = -\frac{1}{2}\left[J_0(x) - J_2(x)\right] = \frac{1}{2}[J_2(x) - J_0(x)]$

Thus we have proved that $J_0''(x) = \frac{1}{2}\left[J_2(x) - J_0(x)\right]$

---

11. *Starting from a suitable recurrence relation show that*

(*a*) $4J_n'' = J_{n-2} - 2J_n + J_{n+2}$

(*b*) $8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}$

>> We have the recurrence relation

$$2J_n' = J_{n-1} - J_{n+1} \qquad \ldots(1)$$

Differentiating w.r.t x we have,

$$2J_n'' = J_{n-1}' - J_{n+1}'.$$

Multiplying by 2 we obtain,

$$4J_n'' = 2J_{n-1}' - 2J_{n+1}'$$

We use (1) in the RHS of this equation replacing n by $(n-1)$ and also by $(n+1)$.

i.e., $4J_n'' = (J_{n-2} - J_n) - (J_n - J_{n+2})$

Thus $\mathbf{4J_n'' = J_{n-2} - 2J_n + J_{n+2}}$ as required.

Differentiating this relation w.r.t. x again we have,

$$4J_n''' = J_{n-2}' - 2J_n' + J_{n+2}'$$

Multiplying by 2 we obtain,

$$8J_n''' = 2J_{n-2}' - 2(2J_n') + 2(J_{n+2}')$$

Now using (1) in the RHS by replacing n by $(n-2)$ and n by $(n+2)$ for the first and third terms along with (1) for the second term we obtain,

$$8J_n''' = (J_{n-3} - J_{n-1}) - 2(J_{n-1} - J_{n+1}) + (J_{n+1} - J_{n+3})$$

Thus $\mathbf{8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}}$ as required.

---

12. *Prove that* $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

>> We have established in the previous problem the result,

$$8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}$$

Putting $n = 0$ in this equation we have,

$$8J_0''' = J_{-3} - 3J_{-1} + 3J_1 - J_3$$

Using $J_{-n} = (-1)^n J_n$ this equation becomes

$$8J_0''' = -J_3 + 3J_1 + 3J_1 - J_3$$

i.e., $8J_0''' = 6J_1 - 2J_3$ or $4J_0''' = 3J_1 - J_3$

But $-J_1 = \frac{d}{dx}(J_0)$ or $J_1 = -J_0'$

Hence we have $4J_0''' = -3J_0' - J_3$

Thus $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

Note : *The result can be obtained independently also.*

---

13. *Show that* $\int J_3(x)\,dx = c - J_2(x) - \frac{2}{x}J_1(x)$

>> Consider the recurrence relation

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow \quad \int x^{-n}J_{n+1}(x)\,dx = -x^{-n}J_n(x) \qquad \ldots(1)$$

Let us write $J_3(x) = x^2\left[x^{-2}J_3(x)\right]$

$$\therefore \quad \int J_3(x)\,dx = \int x^2\left[x^{-2}J_3(x)\right]dx$$

Integrating RHS by parts we get

$$\int J_3(x)\,dx = x^2\int x^{-2}J_3(x)\,dx - \int\left[\int x^{-2}J_3(x)\right]2x\,dx \qquad \ldots(2)$$

From (i) we have when $n = 2$, $\int x^{-2}J_3(x)\,dx = -x^{-2}J_2(x)$

Using this in the RHS of (2) we get,

$$\int J_3(x)\,dx = x^2\left\{-x^{-2}J_2(x)\right\} - \int -\left\{x^{-2}J_2(x)\right\}\cdot 2x\,dx$$

i.e., $$\int J_3(x)\,dx = -J_2(x) + 2\int x^{-1}J_2(x)\,dx \qquad \ldots(3)$$

From (1) we have when $n = 1$, $\int x^{-1}J_2(x)\,dx = -x^{-1}J_1(x)$

Hence (3) becomes,

$$\int J_3(x)\,dx = -J_2(x) + 2\left\{-x^{-1}J_1(x)\right\}$$

Thus $\int J_3(x)\,dx = -J_2(x) - \frac{2}{x}J_1(x) + c$, c being the constant of integration.

---

14. *Show that* $\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x)$

>> We have the recurrence relation

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$\Rightarrow \quad \int_0^x x^n J_{n-1}(x)\,dx = \left[x^n J_n(x)\right]_0^x = x^n J_n(x) - 0$$

Thus $\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x)$

---

15. *Show that* $\int_0^x x^{-n} J_{n+1}(x)\,dx = \dfrac{1}{2^n\,\Gamma(n+1)} - \dfrac{J_n(x)}{x^n}$

>> We have the recurrence relation

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow \quad \int_0^x x^{-n} J_{n+1}(x)\,dx = -\left[x^{-n} J_n(x)\right]_0^x$$

i.e., $\int_0^x x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) + [x^{-n} J_n(x)]_{x=0}$ . . . (1)

The second term in the RHS is an indeterminate form 0/0 and hence we use the concept of limit for evaluating the same.

$$\lim_{x\to 0} \frac{J_n(x)}{x^n} = \lim_{x\to 0} \frac{1}{x^n} \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$= \lim_{x\to 0} \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{n+2r}} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$\lim_{x \to 0} x^{-n} J_n(x) = \lim_{x \to 0} \left\{ \frac{1}{2^n \Gamma(n+1)} - \frac{x^2}{2^{n+2} \Gamma(n+2)} + \frac{x^4}{2^{n+4} \Gamma(n+3) 2!} \cdots \right\}$$

$$= \frac{1}{2^n \Gamma(n+1)} - 0 + 0 - \cdots = \frac{1}{2^n \Gamma(n+1)}$$

$$\therefore \quad \lim_{x \to 0} x^{-n} J_n(x) = \frac{1}{2^n \Gamma(n+1)}$$

Thus by using this result in (1) we have,

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \Gamma(n+1)} - \frac{J_n(x)}{x^n}$$

---

16. *Show that* $\int_0^x x J_0(ax)\, dx = \frac{x}{a} J_1(ax)$

>> We have the recurrence relation : $\frac{d}{dx}\left[x^n J_n(x) \right] = x^n J_{n-1}(x)$

Putting $n = 1$ we have, $\frac{d}{dx}\left[x J_1(x) \right] = x J_0(x)$

Put $x = at$ so that we have

$$\frac{d}{dt}\left[at\, J_1(at) \right] \frac{dt}{dx} = at\, J_0(at)$$

i.e., $$\frac{d}{dt}\left[at\, J_1(at) \right] \frac{1}{a} = at\, J_0(at)$$

or $$\frac{d}{dt}\left[t\, J_1(at) \right] = at\, J_0(at)$$

$$\Rightarrow \quad \int_0^t t\, J_0(at)\, dt = \frac{1}{a}\left[t\, J_1(at) \right]_0^t = \frac{t}{a} J_1(at)$$

Now replacing the variable t by arbitrary variable x we have

$$\int_0^x x J_0(ax)\, dx = \frac{x}{a} J_1(ax)$$

---

17. *Show that* $\frac{d}{dx}\left[J_n^2(x)\right] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$

>> $\frac{d}{dx}\left[J_n^2(x)\right] = 2J_n(x)J_n'(x)$ ...(1)

We have recurrence relations

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

and $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Using these relations in the RHS of (1) we have,

$$\frac{d}{dx}[J_n^2(x)] = 2\cdot\frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]\cdot\frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Thus $\frac{d}{dx}[J_n^2(x)] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$

Prove the following :

18. $xJ_n(x) = 2\left[(n+1)J_{n+1}(x) - (n+3)J_{n+3}(x) + (n+5)J_{n+5}(x) - \cdots\right]$

19. $xJ_{n-1}(x) = 2\left[nJ_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) - \cdots\right]$

20. $xJ_n'(x) = nJ_n(x) - 2\left[(n+2)J_{n+2}(x) - (n+4)J_{n+4}(x) + \cdots\right]$

18. >> We have the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

or $J_n(x) + J_{n+2}(x) = \frac{2(n+1)}{x}J_{n+1}(x)$

$\therefore$ $xJ_n(x) = 2(n+1)J_{n+1}(x) - xJ_{n+2}(x)$ ...(1)

Also $xJ_{n+2}(x) = 2(n+3)J_{n+3}(x) - xJ_{n+4}(x)$

$xJ_{n+4}(x) = 2(n+5)J_{n+5}(x) - xJ_{n+6}(x)$ and so on.

By back substitution (1) becomes

$$xJ_n(x) = 2(n+1)J_{n+1}(x) - 2(n+3)J_{n+3}(x) + 2(n+5)J_{n+5}(x) - \cdots$$

Thus $xJ_n(x) = 2\left[(n+1)J_{n+1}(x) - (n+3)J_{n+3}(x) + (n+5)J_{n+5}(x) - \cdots\right]$

19. >> Again we have from the recurrence relation (1)

$$x J_{n-1}(x) = 2n J_n(x) - x J_{n+1}(x) \quad \ldots (2)$$

Also $\quad x J_{n+1}(x) = 2(n+2) J_{n+2}(x) - x J_{n+3}(x)$

$x J_{n+3}(x) = 2(n+4) J_{n+4}(x) - x J_{n+5}(x)$ and so on.

By back substitution (2) becomes

$$x J_{n-1}(x) = 2n J_n(x) - 2(n+2) J_{n+2}(x) + 2(n+4) J_{n+4}(x) - \cdots$$

Thus $\quad \boldsymbol{x J_{n-1}(x) = 2[n J_n(x) - (n+2) J_{n+2}(x) + (n+4) J_{n+4}(x) - \cdots]}$

20. >> We have recurrence relations

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \ldots (3)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

or $\quad J_{n+1}(x) + J_{n+3}(x) = \dfrac{2(n+2)}{x} J_{n+2}(x)$

or $\quad x J_{n+1}(x) = 2(n+2) J_{n+2}(x) - x J_{n+3}(x) \quad \ldots (4)$

Also $\quad x J_{n+3}(x) = 2(n+4) J_{n+4}(x) - x J_{n+5}(x)$

$x J_{n+5}(x) = 2(n+6) J_{n+6}(x) - x J_{n+7}(x)$ and so on.

By back substitution (3) becomes,

$$x J_n'(x) = n J_n(x) - 2(n+2) J_{n+2}(x) + 2(n+4) J_{n+4}(x) - \cdots$$

Thus $\quad \boldsymbol{x J_n'(x) = n J_n(x) - 2[(n+2) J_{n+2}(x) - (n+4) J_{n+4}(x) + \cdots]}$

---

21. *Verify that $y = x^n J_n(x)$ is a solution of the differential equation*

$$xy'' + (1-2n)y' + xy = 0$$

>> By data $y = x^n J_n(x)$

$\therefore \quad y' = x^n J_n'(x) + n x^{n-1} J_n(x)$

$$y'' = x^n J_n''(x) + 2n x^{n-1} J_n'(x) + n(n-1) x^{n-2} J_n(x)$$

Substituting in $xy'' + (1-2n)y' + xy$ we obtain

$$x\left\{x^n J_n''(x) + 2nx^{n-1} J_n'(x) + n(n-1)x^{n-2} J_n(x)\right\}$$
$$+ (1-2n)\left\{x^n J_n'(x) + nx^{n-1} J_n(x)\right\} + x \cdot x^n J_n(x)$$
$$= x^{n+1} J_n''(x) + 2nx^n J_n'(x) + n^2 x^{n-1} J_n(x) - nx^{n-1} J_n(x)$$
$$+ x^n J_n'(x) + nx^{n-1} J_n(x) - 2nx^n J_n'(x)$$
$$- 2n^2 x^{n-1} J_n(x) + x^{n+1} J_n(x)$$
$$= x^{n+1} J_n''(x) + x^n J_n'(x) + x^{n+1} J_n(x) - n^2 x^{n-1} J_n(x)$$
$$= x^{n-1}\left\{x^2 J_n''(x) + xJ_n'(x) + (x^2-n^2)J_n(x)\right\}$$
$$= x^{n-1} \cdot 0 = 0$$

since $J_n(x)$ is a solution of $x^2 y'' + xy' + (x^2-n^2)y = 0.$

Thus we have proved that $y = x^n J_n(x)$ is a solution of the equation

$$\mathbf{xy'' + (1-2n)y' + xy = 0}$$

---

22. *Verify that $y = \sqrt{x}\, J_{3/2}(x)$ is a solution of the differential equation given by*

$$x^2 y'' + (x^2-2)y = 0$$

>> By data $y = \sqrt{x}\, J_{3/2}(x)$

$$\therefore \quad y' = \sqrt{x}\, J_{3/2}'(x) + J_{3/2}(x) \cdot \frac{1}{2\sqrt{x}}$$

$$y'' = \sqrt{x}\, J_{3/2}''(x) + J_{3/2}'(x) \cdot \frac{1}{2\sqrt{x}} + J_{3/2}(x) \cdot \frac{1}{2} \cdot \frac{-1}{2} x^{-3/2} + \frac{1}{2\sqrt{x}} J_{3/2}'(x)$$

i.e., $$y'' = \sqrt{x}\, J_{3/2}''(x) + \frac{J_{3/2}'(x)}{\sqrt{x}} - \frac{1}{4x\sqrt{x}} J_{3/2}(x)$$

$$\therefore \quad x^2 y'' + (x^2-2)y = \left\{x^{5/2} J_{3/2}''(x) + x^{3/2} J_{3/2}'(x) - \frac{\sqrt{x}}{4} J_{3/2}(x)\right\}$$
$$+ \left\{x^{5/2} J_{3/2}(x) - 2\sqrt{x}\, J_{3/2}(x)\right\}$$
$$= \sqrt{x}\left[x^2 J_{3/2}''(x) + xJ_{3/2}'(x) - \frac{1}{4} J_{3/2}(x) + x^2 J_{3/2}(x) - 2J_{3/2}(x)\right]$$
$$= \sqrt{x}\left\{x^2 J_{3/2}''(x) + xJ_{3/2}'(x) + \left[x^2 - (3/2)^2\right] J_{3/2}(x)\right\}$$

$$= \sqrt{x}\left[x^2 u'' + x u' + (x^2 - n^2) u\right] \text{ where } u = J_{3/2}(x) \text{ and } n = 3/2.$$

Since $J_{3/2}(x)$ is a solution of the Bessel's equation in the standard form, we have

$$x^2 u'' + x u' + (x^2 - n^2) u = 0$$

Thus $x^2 y'' + (x^2 - 2) y = 0$ **as required.**

---

23. *Show by the substitution* $u = 2n\sqrt{x}$ *that the differential equation* $x\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} + n^2 y = 0$ *is transformed into Bessel's equation of order zero. Hence show that* $y = AJ_0(2n\sqrt{x})$ *is a general solution of the equation.*

>> We have, $u^2 = 4n^2 x$ since $u = 2n\sqrt{x}$ by data.

Differentiating w.r.t. x we have,

$$2u\frac{du}{dx} = 4n^2 \text{ or } \frac{du}{dx} = \frac{2n^2}{u}$$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{dy}{du} \cdot \dfrac{2n^2}{u}$

Differentiating this again w.r.t. x we have,

$$\frac{d^2 y}{dx^2} = 2n^2\left[\frac{dy}{du}\frac{-1}{u^2}\frac{du}{dx} + \frac{1}{u}\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]$$

$$= 2n^2\left[\frac{-2n^2}{u^3}\frac{dy}{du} + \frac{2n^2}{u^2}\frac{d^2 y}{du^2}\right]$$

i.e., $$\frac{d^2 y}{dx^2} = \frac{4n^4}{u^2}\left[\frac{-1}{u}\frac{dy}{du} + \frac{d^2 y}{du^2}\right]$$

Consider $x\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} + n^2 y = 0$, where $x = \dfrac{u^2}{4n^2}$

i.e., $$\frac{u^2}{4n^2} \cdot \frac{4n^4}{u^2}\left[\frac{-1}{u}\frac{dy}{du} + \frac{d^2 y}{du^2}\right] + \frac{2n^2}{u}\frac{dy}{du} + n^2 y = 0$$

i.e., $$\frac{-n^2}{u}\frac{dy}{du} + n^2\frac{d^2 y}{du^2} + \frac{2n^2}{u}\frac{dy}{du} + n^2 y = 0$$

i.e., $$n^2\left[\frac{d^2 y}{du^2} + \frac{1}{u}\frac{dy}{du} + y\right] = 0 \quad \text{or} \quad \frac{d^2 y}{du^2} + \frac{1}{u}\frac{dy}{du} + y = 0$$

Multiplying by u^2 we get $u^2 \frac{d^2 y}{du^2} + u \frac{dy}{du} + u^2 y = 0$

i.e., $u^2 \frac{d^2 y}{du^2} + u \frac{dy}{du} + (u^2 - 0^2) y = 0$

This is Bessel's equation of order zero and its general solution is

$y = a J_n(u) + b J_{-n}(u)$ where $n = 0$

i.e., $y = a J_0(2n\sqrt{x}) + b J_0(2n\sqrt{x}) = (a+b) J_0(2n\sqrt{x})$

Thus by denoting $A = a+b$, we conclude that

$y = A J_0(2n\sqrt{x})$ **is a general solution of the given equation.**

---

24. *Prove that* $\frac{d}{dx}\left[x\left\{J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x)\right\}\right] = 0$

>> We know that $J_n(x)$ and $J_{-n}(x)$ are solutions of the Bessel's equation

$x^2 y'' + x y' + (x^2 - n^2) y = 0$

If $u = J_n(x)$ and $v = J_{-n}(x)$ we have,

$x^2 u'' + x u' + (x^2 - n^2) u = 0$ ...(1)

and $x^2 v'' + x v' + (x^2 - n^2) v = 0$ ...(2)

Multiplying (1) by v and (2) by u we have,

$x^2 v u'' + x v u' + x^2 u v - n^2 u v = 0$

and $x^2 v'' u + x v' u + x^2 u v - n^2 u v = 0$

On subtracting and dividing by x we obtain,

$x(v u'' - v'' u) + (v u' - v' u) = 0$

or $\frac{d}{dx}\left[x(v u' - v' u)\right] = 0$

Thus $\frac{d}{dx}\left[x\left\{J_{-n}(x)J_n'(x) - J_{-n}'(x)J_n(x)\right\}\right] = 0$

---

25. *Prove that*

$$x\left[J_n'(x)J_{-n}(x)-J_{-n}'(x)J_n(x)\right]=\frac{2\sin n\pi}{\pi}$$

>> $$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_{-n}(x)=\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{-n+2s}\frac{1}{\Gamma(-n+s+1)\,s!}$$

$\therefore$ $$J_n'(x)=\sum_{r=0}^{\infty}(-1)^r(n+2r)\left(\frac{x}{2}\right)^{n+2r-1}\cdot\frac{1}{2}\cdot\frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_{-n}'(x)=\sum_{s=0}^{\infty}(-1)^s(-n+2s)\left(\frac{x}{2}\right)^{-n+2s-1}\cdot\frac{1}{2}\cdot\frac{1}{\Gamma(-n+s+1)\,s!}$$

Consider $x\,[J_n'(x)J_{-n}(x)-J_{-n}'(x)J_n(x)]$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\left(\frac{x}{2}\right)^{2(r+s)}\frac{(n+2r)}{\Gamma(-n+s+1)\Gamma(n+r+1)\,r!\,s!}$$

$$-\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\left(\frac{x}{2}\right)^{2(r+s)}\frac{(-n+2s)}{\Gamma(-n+s+1)\Gamma(n+r+1)\,r!\,s!}$$

It can be easily seen that on expansion all the terms cancel out except the terms when $r=0$ and $s=0$. Hence we have,

$$\frac{n}{\Gamma(-n+1)\Gamma(n+1)}-\frac{-n}{\Gamma(-n+1)\Gamma(n+1)}$$

$$=\frac{n}{\Gamma(1-n)\,n\,\Gamma(n)}+\frac{n}{\Gamma(1-n)\,n\,\Gamma(n)}=\frac{2}{\Gamma(1-n)\Gamma(n)}$$

But we know that $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$

Thus $x\,[J_n'(x)J_{-n}(x)-J_{-n}'(x)J_n(x)]=\dfrac{2\sin n\pi}{\pi}$

Note : *Alternative version of the problem. Show th·t* $\dfrac{d}{dx}\left[\dfrac{J_{-n}(x)}{J_n(x)}\right]=\dfrac{-2\sin n\pi}{\pi x J_n^2(x)}$

---

26. *Obtain the solution of the following equation in terms of Bessel functions.*

$$\frac{d^2 y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{1}{9x^2}\right)y = 0$$

>> Multiplying the given equation by x^2 we have,

$$x^2 y''+xy'+(x^2-1/9)y = 0 \qquad \ldots(1)$$

We have Bessel's equation in the form

$$x^2 y''+xy'+(x^2-n^2)y = 0 \qquad \ldots(2)$$

The solution is given by $y = a J_n(x)+b J_{-n}(x)$

Comparing (1) and (2) we have $n^2 = 1/9$ or $n = \pm 1/3$

Thus the solution of the given equation in terms of Bessel functions is

$$y = aJ_{1/3}(x)+bJ_{-1/3}(x)$$

---

27. *Solve* : $16x^2 y''+16xy'+(16x^2-1)y = 0$ *in terms of Bessel functions.*

>> The given equation on dividing by 16 becomes,

$$x^2 y''+xy'+(x^2-1/16)y = 0 \qquad \ldots(1)$$

We have Bessel's equation in the form,

$$x^2 y''+xy'+(x^2-n^2)y = 0 \qquad \ldots(2)$$

The solution is given by $y = a J_n(x)+bJ_{-n}(x)$

Comparing (1) and (2) we have, $n^2 = 1/16$ or $n = \pm 1/4$.

Thus the solution of the given equation in terms of Bessel functions is,

$$y = aJ_{1/4}(x)+bJ_{-1/4}(x)$$

---

28. *Solve* : $y'' + \frac{y'}{x} + \left(1 - \frac{1}{6.25x^2}\right) y = 0$ *in terms of Bessel functions.*

>> Multiplying the given equation by x^2 and writing 6.25 as $25/4 = (5/2)^2$ we have,

$$x^2 y'' + xy' + [x^2 - (2/5)^2] y = 0$$

This problem is similiar to the previous one where $n = 2/5$.

Thus the solution of the given equation in terms of Besel functions is,

$$y = a J_{2/5}(x) + b J_{-2/5}(x)$$

---

29. *Show that the solution of the equation*

$$x^2 y'' + x y' + (x^2 - 1/4) y = 0 \quad is \quad c_1 \frac{\sin x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}}$$

>> Comparing the given equation with Bessel's equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0, \text{ we have } n^2 = 1/4 \text{ or } n = \pm 1/2$$

The solution of the given equation in terms of Bessel functions is

$$y = a J_{1/2}(x) + b J_{-1/2}(x)$$

But $J_{1/2}(x) = \sqrt{2/\pi x} \sin x$ and $J_{-1/2}(x) = \sqrt{2/\pi x} \cos x$ [*Refer Problem* - 1]

Hence $y = a\sqrt{2/\pi x} \sin x + b\sqrt{2/\pi x} \cos x$

ie., $y = \left(a\sqrt{2/\pi}\right) \frac{\sin x}{\sqrt{x}} + (b\sqrt{2/\pi}) \frac{\cos x}{\sqrt{x}}$

Let $c_1 = a\sqrt{2/\pi}$ and $c_2 = b\sqrt{2/\pi}$; c_1 and c_2 are arbitrary constants.

Thus $y = c_1 \frac{\sin x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}}$

5.44 Orthogonal Property of Bessel Functions

If α *and* β *are two distinct roots of* $J_n(x) = 0$ *then*

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \begin{cases} 0, \text{ if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2}[J_{n+1}(\alpha)]^2 \text{ if } \alpha = \beta \end{cases}$$

Proof : We know that $J_n(\lambda x)$ is a solution of the equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0 \qquad [\textit{Article } 5.41]$$

If $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ the associated differential equations are

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \qquad \ldots (1)$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \qquad \ldots (2)$$

Multiplying (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$ we obtain,

$$x v u'' + v u' + \alpha^2 u v x - \frac{n^2 u v}{x} = 0$$

and $$x u v'' + u v' + \beta^2 u v x - \frac{n^2 u v}{x} = 0$$

On subtracting we obtain

$$x(v u'' - u v'') + (v u' - u v') + (\alpha^2 - \beta^2) u v x = 0$$

i.e., $$\frac{d}{dx}\{x(v u' - u v')\} = (\beta^2 - \alpha^2) x u v$$

Integrating both sides w.r.t. x between 0 and 1 we have

$$\left[x(v u' - u v')\right]_{x=0}^{1} = (\beta^2 - \alpha^2)\int_{x=0}^{1} x u v\, dx$$

i.e., $$(v u' - u v')_{x=1} - 0 = (\beta^2 - \alpha^2)\int_0^1 x u v\, dx \qquad \ldots (3)$$

Since $u = J_n(\alpha x)$, $v = J_n(\beta x)$ we have $u' = \alpha J_n'(\alpha x)$ $v' = \beta J_n'(\beta x)$ and as a consequence of these (3) becomes

$$\left[J_n(\beta x)\alpha J_n'(\alpha x) - J_n(\alpha x)\beta J_n'(\beta x)\right]_{x=1} = (\beta^2 - \alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx$$

Hence $\displaystyle\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\,dx = \frac{1}{\beta^2 - \alpha^2}\left\{\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right\}$ ...(4)

Since α & β are distinct roots of $J_n(x) = 0$ we have $J_n(\alpha) = 0$ & $J_n(\beta) = 0$, with the result the RHS of (4) becomes zero provided $\beta^2 - \alpha^2 \neq 0$ or $\beta \neq \alpha$.

Thus we have proved that if $\alpha \neq \beta$,

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\,dx = 0 \qquad \text{...(5)}$$

We shall now discuss the case when $\alpha = \beta$

The RHS of (3) becomes an indeterminate form of the type $\frac{0}{0}$ when $\alpha = \beta$.

We shall evaluate by taking limits on both sides as $\beta \to \alpha$, keeping α fixed, by applying L' Hospital's rule.

i.e., $\displaystyle\lim_{\beta\to\alpha}\int_0^1 x J_n(\alpha x)\, J_n(\beta x)\,dx$

$$= \lim_{\beta\to\alpha}\frac{1}{\beta^2 - \alpha^2}\left\{\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)\right\}$$

Since α is fixed we must have $J_n(\alpha) = 0$ as α is a root of $J_n(x) = 0$

$$\therefore \quad \lim_{\beta\to\alpha}\int_0^1 x J_n(\alpha x)\, J_n(\beta x)\,dx = \lim_{\beta\to\alpha}\frac{1}{\beta^2 - \alpha^2}\left\{\alpha J_n(\beta) J_n'(\alpha)\right\}$$

$$= \lim_{\beta\to\alpha}\frac{1}{2\beta}\left\{\alpha J_n'(\beta)\, J_n'(\alpha)\right\} \text{ by L' Hospital's rule.}$$

The numerator and denominator are differentiated separately w.r.t. β

We now have,

$$\int_0^1 x\,[J_n(\alpha x)]^2\,dx = \frac{1}{2\alpha}\,\alpha\,J_n'(\alpha)J_n'(\alpha) = \frac{1}{2}\,[J_n'(\alpha)]^2$$

$$\therefore \quad \int_0^1 x\,J_n^2(\alpha x)\,dx = \frac{1}{2}\,[J_n'(\alpha)]^2 \qquad \ldots(6)$$

Further we have the recurrence relation $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$

$$\therefore \quad J_n'(\alpha) = \frac{n}{\alpha}J_n(\alpha) - J_{n+1}(\alpha)$$

Since $J_n(\alpha) = 0$, we obtain $J_n'(\alpha) = -J_{n+1}(\alpha)$ and (6) now becomes,

$$\int_0^1 x\,J_n^2(\alpha x)\,dx = \frac{1}{2}\,[J_{n+1}(\alpha)]^2$$

This result is known as the ***Lommel integral formula.***

Note : *The orthogonal property is also presented in the form :*

$$\int_0^a x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{a^2}{2}\,[J_{n+1}(a\alpha)]^2 & \text{if } \alpha = \beta \end{cases}$$

This result can be established working on similar lines as before.

5.5 Series Solution of Legendre's Differential Equation

We have Legendre differential equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \ldots(1)$$

The coefficient of $y'' = (1-x^2) = P_0(x)$ and $P_0(x) \neq 0$ at $x = 0$.

We employ power series method to solve this equation.

We assume the series solution of (1) in the form

$$y = \sum_{r=0}^{\infty} a_r\,x^r \qquad \ldots(2)$$

$$\therefore \quad \frac{dy}{dx} = \sum_0^\infty a_r\, r x^{r-1}, \quad \frac{d^2y}{dx^2} = \sum_0^\infty a_r\, r(r-1)x^{r-2}$$

Now (1) becomes,

$$(1-x^2)\sum_0^\infty a_r r(r-1)x^{r-2} - 2x\sum_0^\infty a_r r x^{r-1} + n(n+1)\sum_0^\infty a_r x^r = 0$$

i.e., $$\sum_0^\infty a_r r(r-1)x^{r-2} - \sum_0^\infty a_r r(r-1)x^r - \sum_0^\infty 2a_r r x^r + n(n+1)\sum_0^\infty a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2} and x^{-1} available only in the first summation to zero.

Coeff. of x^{-2}: $\quad a_0(0)(-1) = 0 \Rightarrow \quad a_0 \neq 0$

Coeff. of x^{-1}: $\quad a_1(1)(0) = 0 \Rightarrow \quad a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero.

i.e., $$a_{r+2}(r+2)(r+1) - a_r r(r-1) - 2a_r r + n(n+1)a_r = 0$$

i.e., $$a_{r+2}(r+2)(r+1) = a_r[r(r-1)+2r-n(n+1)]$$

or $$a_{r+2} = \frac{-[n(n+1)-r^2-r]}{(r+2)(r+1)}\, a_r \qquad \dots(3)$$

Putting $r = 0, 1, 2, 3, \dots$ in (3) we obtain,

$$a_2 = \frac{-n(n+1)}{2}a_0\ ;\ a_3 = \frac{-(n^2+n-2)}{6}a_1 = \frac{-(n-1)(n+2)}{6}a_1$$

$$a_4 = \frac{-(n^2+n-6)}{12}\cdot a_2 = \frac{-(n-2)(n+3)}{12}\cdot\frac{-n(n+1)}{2}a_0$$

i.e., $$a_4 = \frac{n(n+1)(n-2)(n+3)}{24}a_0$$

$$a_5 = \frac{-(n^2+n-12)}{20}\cdot a_3 = \frac{-(n-3)(n+4)}{20}\cdot\frac{-(n-1)(n+2)}{6}a_1$$

i.e., $$a_5 = \frac{(n-1)(n+2)(n-3)(n+4)}{120}a_1 \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

Since constants $a_2, a_4, \ldots$ are in terms of a_0 and $a_3, a_5, \ldots$ are in terms of a_1 we rearrange the RHS in the form

$$y = (a_0 + a_2 x + a_4 x^2 + \cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$$

On substituting for $a_2, a_3, a_4, a_5 \ldots$ we obtain

$$y = a_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4 - \cdots \right]$$

$$+ a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} x^5 - \cdots \right] \quad \ldots (4)$$

Let $u(x)$ & $v(x)$ respectively represent the two infinite series in (4) so that we have

$$y = a_0 u(x) + a_1 v(x) \quad \ldots (5)$$

This is the series solution of Legendre's differential equation.

5.51 Legendre Polynomials

If n is a positive even integer, $a_0 u(x)$ reduces to a polynomial of degree n and if n is a positive odd integer $a_1 v(x)$ reduces to a polynomial of degree n. Otherwise these will give infinite series called ***Legendre functions of second kind.***

It may be observed that the polynomials $u(x)$, $v(x)$ contain alternate powers of x and a general form of the polynomial that represents either of them in descending powers of x can be presented in the form

$$y = f(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots + F(x) \quad \ldots (1)$$

$$\text{where } F(x) = \begin{cases} a_0 \text{ if } n \text{ is even} \\ a_1 x \text{ if } n \text{ is odd} \end{cases}$$

We note that a_r is the coefficient of x^r in the series solution of the differential equation and we have obtained, [*Refer (3) in the previous article*]

$$a_{r+2} = \frac{-[n(n+1) - r(r+1)]}{(r+2)(r+1)} a_r \quad \ldots (2)$$

We plan to express $a_{n-2}, a_{n-4}, \ldots$ present in (1) in terms of a_n. Replacing r by $(n-2)$ in (2) we obtain

$$a_n = \frac{-[n(n+1)-(n-2)(n-1)]}{n(n-1)} a_{n-2}$$

i.e., $$a_n = \frac{-(4n-2)}{n(n-1)} a_{n-2}$$

or $$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

Again from (2), on replacing r by $(n-4)$ we obtain

$$a_{n-2} = \frac{-[n(n+1)-(n-4)(n-3)]}{(n-2)(n-3)} a_{n-4}$$

i.e., $$a_{n-2} = \frac{-(8n-12)}{(n-2)(n-3)} a_{n-4}$$

or $$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$\therefore$ $a_{n-4} = \dfrac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} a_n$ by using the value of a_{n-2} and so-on.

Using these values in (1) we have,

$$y = f(x) = a_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} x^{n-4} - \cdots + G(x) \right]$$

where $G(x) = \begin{cases} a_0/a_n & \text{if } n \text{ is even.} \\ a_1 x/a_n & \text{if } n \text{ is odd.} \end{cases}$

If the constant a_n is so chosen such that $y = f(x)$ becomes 1 when $x = 1$, the polynomials so obtained are called Legendre polynomials denoted by $P_n(x)$.

Let us choose $a_n = \dfrac{1\cdot 3\cdot 5\cdots 2n-1}{n!}$ to meet the said requirement. That is

$$P_n(x) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} x^{n-4} - \cdots \right] \quad \ldots (3)$$

We obtain first few Legendre polynomials by putting $n = 0, 1, 2, 3, 4$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{1!}[x] = x$$

$$P_2(x) = \frac{1\cdot 3}{2!}\left[x^2 - \frac{2(2-1)}{2\cdot 3}x^0\right] = \frac{3}{2}\left(x^2 - \frac{1}{3}\right) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1\cdot 3\cdot 5}{3!}\left[x^3 - \frac{3(2)}{2\cdot 5}x\right] = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1\cdot 3\cdot 5\cdot 7}{4!}\left[x^4 - \frac{4(3)}{2(7)}x^2 + \frac{4(3)(2)(1)}{2\cdot 4(7)(5)}\right]$$

i.e., $$P_4(x) = \frac{35}{8}\left[x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right] = \frac{1}{8}[35x^4 - 30x^2 + 3] \text{ etc.}$$

It can be easily seen that all these expressions give 1 at $x = 1$ in accordence with the definition of Legendre polynomials.

5.52 Rodrigue's formula

We derive a formula for the Legendre polynomials $P_n(x)$ in the form

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$ *known as* ***Rodrigue's formula.***

Proof: Let $u = (x^2 - 1)^n$

We shall first establish that the n^{th} derivative of u, that is u_n is a solution of the Legendre's differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots(1)$$

Differentiating u w.r.t. x, we have,

$$\frac{du}{dx} = u_1 = n(x^2 - 1)^{n-1}\cdot 2x \quad \text{or} \quad (x^2 - 1)u_1 = 2nx(x^2 - 1)^n$$

i.e., $$(x^2 - 1)u_1 = 2nxu$$

Differentiating w.r.t. x again we have,

$$(x^2 - 1)u_2 + 2xu_1 = 2n(xu_1 + u)$$

We shall now differentiate the result n times by applying Leibnitz theorem for the n^{th} derivative of a product given by

$$(UV)_n = UV_n + nU_1V_{n-1} + \frac{n(n-1)}{2!}U_2V_{n-2} + \cdots + U_nV$$

$$\therefore \quad [(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$$

ie.,

$$\left[(x^2-1)u_{n+2} + n\cdot 2x\cdot u_{n+1} + \frac{n(n-1)}{2}\cdot 2\cdot u_n\right] + 2\left[xu_{n+1} + n\cdot 1\cdot u_n\right]$$

$$= 2n\left[xu_{n+1} + n\cdot 1\cdot u_n\right] + 2nu_n$$

i.e., $(x^2-1)u_{n+2} + 2nxu_{n+1} + (n^2-n)u_n + 2xu_{n+1} + 2nu_n$

$$= 2nxu_{n+1} + 2n^2u_n + 2nu_n$$

i.e., $(x^2-1)u_{n+2} + 2xu_{n+1} - n^2u_n - nu_n = 0$

i.e., $(x^2-1)u_{n+2} + 2xu_{n+1} - nu_n(n+1) = 0$

or $(1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0$

This can be put in the form,

$$(1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0 \qquad \ldots (2)$$

Comparing (2) with (1) we conclude that u_n is a solution of the Legendre's differential equation. It may be observed that u is a polynomial of degree $2n$ and hence u_n will be a polynomial of degree n.

Also $P_n(x)$ which satisfies the Legendre differential equation is also a polynomial of degree n. Hence u_n must be the same as $P_n(x)$ but for some constant factor k.

i.e., $P_n(x) = ku_n = k[(x^2-1)^n]_n$

i.e., $P_n(x) = k[(x-1)^n(x+1)^n]_n$

Applying Leibnitz theorem for the RHS we have,

$$P_n(x) = k\Big[(x-1)^n \{(x+1)^n\}_n + n \cdot n (x-1)^{n-1} \{(x+1)^n\}_{n-1} + \frac{n(n-1)}{2!} n(n-1)(x-1)^{n-2} \{(x+1)^n\}_{n-2} + \cdots \{(x-1)^n\}_n (x+1)^n \Big] \quad \ldots (3)$$

It should be observed that if $Z = (x-1)^n$ then

$$Z_1 = n(x-1)^{n-1}, \; Z_2 = n(n-1)(x-1)^{n-2} \text{ etc.}$$

$$Z_n = n(n-1)(n-2)\cdots 2 \cdot 1 (x-1)^{n-n} \text{ or } Z_n = n!(x-1)^0 = n!$$

$$\therefore \quad \{(x-1)^n\}_n = n!$$

We proceed to find k by choosing a suitable value for x.
Putting $x = 1$ in (3) all the terms in RHS become zero except the last term which becomes $n!(1+1)^n = n!2^n$.

i.e., $P_n(1) = k \cdot n! \cdot 2^n$ and $P_n(1) = 1$ by the definition of $P_n(x)$.

$$\therefore \quad 1 = k \cdot n! \cdot 2^n \quad \text{or} \quad k = \frac{1}{n!\,2^n}$$

Since $P_n(x) = k u_n$, we have, $P_n(x) = \dfrac{1}{n!\,2^n}\{(x^2-1)^n\}_n$

Thus we have proved that $P_n(x) = \dfrac{1}{2^n \cdot n!} \dfrac{d^n}{dx^n}(x^2-1)^n$ [***Rodrigue's Formula***]

WORKED PROBLEMS

30. *Using Rodrigue's formula obtain expressions for $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$. Hence express x^2, x^3, x^4, x^5 in terms of Legendre polynomials.*

>> We have Rodrigue's formula

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2-1)^n$$

We shall put $n = 0, 1, 2, 3, 4, 5$ successively in this formula.

$$P_0(x) = \frac{1}{2^0\, 0!} \frac{d^0}{dx^0}(x^2-1)^0 = 1$$

$$\therefore \quad \mathbf{P_0(x) = 1}$$

$$P_1(x) = \frac{1}{2^1\,1!}\,\frac{d}{dx}\,(x^2-1) = \frac{1}{2}(2x) = x$$

$\therefore \quad P_1(x) = x$

$$P_2(x) = \frac{1}{2^2\,2!}\,\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{8}\,\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{8}\,(12x^2-4) = \frac{1}{2}(3x^2-1)$$

$\therefore \quad P_2(x) = \frac{1}{2}(3x^2-1)$

$$P_3(x) = \frac{1}{2^3\,3!}\,\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\,\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$$

$$P_3(x) = \frac{1}{48}\,(120x^3 - 72x) = \frac{24}{48}\,(5x^3 - 3x)$$

$\therefore \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$P_4(x) = \frac{1}{2^4\,4!}\,\frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{16\times 24}\,\frac{d^4}{dx^4}(x^8-4x^6+6x^4-4x^2+1)$$

where we have used the binomial expansion

$$(x-a)^n = x^n - {}^nC_1\,x^{n-1}a + {}^nC_2\,x^{n-2}a^2 - {}^nC_3\,x^{n-3}a^3 + \cdots + (-1)^n a^n$$

We shall also use, $\dfrac{d^n}{dx^n}\,(x^m) = \dfrac{m!}{(m-n)!}\,x^{m-n}$, where $m > n$

Now, $$P_4(x) = \frac{1}{16\times 24}\left[\frac{8!}{4!}\,x^4 - 4\times\frac{6!}{2!}\,x^2 + 6\times\frac{4!}{0!}\,x^0\right]$$

$$= \frac{1}{16\times 24}\,[1680x^4 - 1440x^2 + 144]$$

$$= \frac{48}{16\times 24}\,[35x^4 - 30x^2 + 3]$$

$\therefore \quad P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

$$P_5(x) = \frac{1}{2^5 5!}\frac{d^5}{dx^5}(x^2-1)^5$$

$$= \frac{1}{32\times 120}\frac{d^5}{dx^5}(x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1)$$

$$= \frac{1}{32\times 120}\left(\frac{10!}{5!}x^5 - 5\times\frac{8!}{3!}x^3 + 10\times\frac{6!}{1!}x\right)$$

$$= \frac{1}{32\times 120}(30240x^5 - 33600x^3 + 7200x)$$

$\therefore \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$, on simplification.

We now express x^2, x^3, x^4, x^5 in terms of Legendre polynomials.

Consider $P_2(x) = \frac{1}{2}(3x^2 - 1)$

or $\quad 2P_2(x) = 3x^2 - 1$ or $3x^2 = 1 + 2P_2(x)$. But $P_0(x) = 1$

$\therefore \quad x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$ ...(1)

Next consider $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

or $\quad 2P_3(x) = 5x^3 - 3x$ or $5x^3 = 2P_3(x) + 3x$. But $P_1(x) = x$

$\therefore \quad x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$ ...(2)

Now consider $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

i.e., $\quad 8P_4(x) = 35x^4 - 30x^2 + 3$ or $35x^4 = 8P_4(x) + 30x^2 - 3$

i.e., $\quad 35x^4 = 8P_4(x) + 10[2P_2(x) + 1] - 3$, by using (1).

i.e., $\quad 35x^4 = 8P_4(x) + 20P_2(x) + 7$

$\therefore \quad x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$ ...(3)

Next consider $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

i.e., $8P_5(x) = 63x^5 - 70x^3 + 15x$ or $63x^5 = 8P_5(x) + 70x^3 - 15x$

Using $x = P_1(x)$ and (2) we have,

$$63x^5 = 8P_5(x) + 14\,[2P_3(x) + 3P_1(x)] - 15P_1(x)$$

i.e., $63x^5 = 8P_5(x) + 28P_3(x) + 27P_1(x)$

$$\therefore \quad x^5 = \frac{8}{63}P_5(x) + \frac{4}{9}P_3(x) + \frac{3}{7}P_1(x) \qquad \ldots (4)$$

Note : *Expressing $x^2, x^3, x^4, \cdots$ in terms of Legendre polynomials helps us to express any given polynomial $f(x)$ in terms of Legendre polynomials.*

Working procedure for problems

- We write down from memory the expression for $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)\cdots$ in correlation with the degree of the given polynomial.
- We express, $x^2, x^3, x^4, \cdots$ in terms of Legendre polynomials.
- We substitute these expressions in the given polynomial function $f(x)$ and simplify to obtain $f(x)$ in the form : $a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + a_3P_3(x) + \cdots$ where $a_0, a_1, a_2, a_3\cdots$ are constants.

31. *Express $x^3 + 2x^2 - 4x + 5$ in terms of Legendre polynomials.*

>> Let $f(x) = x^3 + 2x^2 - 4x + 5$

We have $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Hence $x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$; $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

Substituting these in $f(x)$ along with $x = P_1(x)$ and $1 = P_0(x)$ we have,

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 2\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right] - 4P_1(x) + 5P_0(x)$$

Thus $f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{17}{5}P_1(x) + \frac{17}{3}P_0(x)$

---

32. *If* $x^3 + 2x^2 - x + 1 = a\,P_0(x) + b\,P_1(x) + c\,P_2(x) + d\,P_3(x)$
find the values of a, b, c, d

>> Let $f(x) = x^3 + 2x^2 - x + 1$

As in the previous example substituting for x^3, x^2, x, 1 in terms of Legendre polynomials we have,

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 2\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right] - P_1(x) + P_0(x)$$

i.e., $$f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) + \frac{5}{3}P_0(x)$$

Hence we have,

$$a\,P_0(x) + b\,P_1(x) + c\,P_2(x) + d\,P_3(x)$$

$$= \frac{5}{3}P_0(x) - \frac{2}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{5}P_3(x)$$

Thus by comparing both sides we obtain

$$a = \frac{5}{3},\ b = -\frac{2}{5},\ c = \frac{4}{3},\ d = \frac{2}{5}$$

---

33. *Show that* $x^4 - 3x^2 + x = \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x)$

>> Let $f(x) = x^4 - 3x^2 + x$ and we have obtained,

$$x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x),\quad x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x).$$

Substituting these in $f(x)$ with $x = P_1(x)$ we have,

$$f(x) = \left[\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)\right] - 3\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right] + P_1(x)$$

Thus $$f(x) = \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x)$$

---

34. *Show that*

(i) $P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$ (ii) $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

>> (i) We have $P_2(x) = \frac{1}{2}(3x^2 - 1)$

Now $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$.

But $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$\therefore \quad P_2(\cos\theta) = \frac{1}{2}\left[\frac{3}{2}(1 + \cos 2\theta) - 1\right] = \frac{1}{4}[3 + 3\cos 2\theta - 2]$$

Thus $\mathbf{P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)}$

(ii) We also have $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Now $P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

But $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$

$$\therefore \quad P_3(\cos\theta) = \frac{1}{2}\left[5 \cdot \frac{1}{4}(\cos 3\theta + 3\cos\theta) - 3\cos\theta\right]$$

$$= \frac{1}{8}[5\cos 3\theta + 15\cos\theta - 12\cos\theta]$$

Thus $\mathbf{P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)}$

---

35. *Obtain $P_3(x)$ from Rodrigue's formula and verify that the same satisfies the Legendre's equation in the standard form.*

>> $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ [*Refer problem-29*]

We have Legendre's equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

We have to verify that,

$$(1-x^2)\,P_3''(x)-2x\,P_3'(x)+3(3+1)\,P_3(x) = 0, \text{ since } n = 3$$

From the expression of $P_3(x)$ we obtain,

$$P_3'(x) = \frac{1}{2}(15x^2-3) \text{ and } P_3''(x) = 15x$$

Consider $(1-x^2)\,P_3''(x)-2x\,P_3'(x)+12\,P_3(x)$

$$= (1-x^2)(15x)-2x\cdot\frac{1}{2}(15x^2-3)+12\cdot\frac{1}{2}(5x^3-3x)$$

$$= 15x-15x^3-15x^3+3x+30x^3-18x = 0$$

Thus we have verified that $P_3(x)$ satisfies Legendre's equation.

---

36. *State Rodrigue's formula for Legendre polynomials and obtain the expression for $P_4(x)$ from it. Verify the property of Legendre polynomials in respect of $P_4(x)$ and also find* $\int_{-1}^{+1} x^3 P_4(x)\,dx$

>> Rodrigue's formula : $P_n(x) = \dfrac{1}{2^n\,n!}\dfrac{d^n}{dx^n}[(x^2-1)^n]$

By putting $n = 4$ we obtain $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$ [*Refer problem - 29*]

Now $P_4(1) = \frac{1}{8}(35-30+3) = 1$

The property $P_n(x) = 1$ in respect of Legendre polynomials is satisfied for $P_4(x)$.

Also, $\int_{-1}^{+1} x^3 P_4(x)\, dx = \int_{-1}^{+1} x^3 \cdot \frac{1}{8}(35x^4 - 30x^2 + 3)\, dx$

$$= \frac{1}{8}\int_{-1}^{+1} (35x^7 - 30x^5 + 3x^3)\, dx$$

$$= \frac{1}{8}\left\{ 35\left[\frac{x^8}{8}\right]_{-1}^{1} - 30\left[\frac{x^6}{6}\right]_{-1}^{1} + 3\left[\frac{x^4}{4}\right]_{-1}^{1} \right\}$$

$$= \frac{1}{8}\left\{ \frac{35}{8}(1-1) - \frac{1}{5}(1-1) + \frac{3}{4}(1-1) \right\} = 0$$

Thus $\int_{-1}^{+1} x^3 P_4(x)\, dx = 0$

EXERCISES

1. Verify that $y = x J_n(x)$ is a solution of the d.e $x^2 y'' - xy' + (1 + x^2 - n^2) y = 0$
2. Verify that $y = J_0(2\sqrt{ax})$ satisfies the d.e $x y'' + y' + a y = 0$
3. Show that $J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$
4. Show that $J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) J_1(x) - \left(\frac{192}{x^3} - \frac{12}{x}\right) J_0(x)$

Express the following polynomials interms of Legendre polynomials

5. $x^3 + x^2 + x + 1$
6. $4x^3 - 2x^2 - 3x + 8$
7. $x^4 + 3x^3 - x^2 + 5x - 2$
8. $(x+1)(x+2)(x+3)$
9. Show that $P_4(\cos\theta) = \frac{1}{64}(35\cos 4\theta + 20\cos\theta + 9)$
10. Obtain the expression for $P_5(x)$ from Rodrigue's formula and verify that the same satisfies the associated Legendre's equation. Also show that $\int_{-1}^{+1} x\, P_5(x)\, dx = 0$

ANSWERS

5. $\frac{4}{3}P_0(x)+\frac{8}{5}P_1(x)+\frac{2}{3}P_2(x)+\frac{2}{5}P_3(x)$

6. $\frac{22}{3}P_0(x)-\frac{3}{5}P_1(x)-\frac{4}{3}P_2(x)+\frac{8}{5}P_0(x)$

7. $\frac{-224}{105}P_0(x)+\frac{34}{5}P_1(x)-\frac{2}{21}P_2(x)+\frac{6}{5}P_2(x)+\frac{8}{35}P_4(x)$

8. $8P_0(x)+\frac{58}{5}P_1(x)+4P_2(x)+\frac{2}{5}P_3(x)$

PART B

Unit - VI

Probability Theory - 1

6.1 Introduction

In our normal conversations we make statements like 'I am likely to be out of station', 'I have a good chance of being selected to the job' etc. In such statements we are not sure of the outcome but we intend to estimate the chances of our statments being true.

Probablity gives an insight for such things in a mathematical way.

6.2 Definitions

Exhaustive event : An event consisting of all the various possibilities is called an *exhaustive event*.

Mutually exclusive events : Two or more events are said to be *mutually exclusive* if the happening of one event prevent the simultaneous happening of the others.

Examples

1. In tossing a coin, getting head and tail are mutually exclusive inview of the fact that if head is the turn out, getting tail is not possible.
2. In throwing a cubical 'die', getting any of the number 1, 2, 3, 4, 5, 6 are mutually exclusive as the turn out of any number rules out the possibility of the turn out of other numbers.

Independent events : Two or more events are said to be *independent* if the happening or nonhappening of one event does not prevent the happening or non happening of the others.

Examples

1. When two coins are tossed the event of getting head is an independent event as both the coins can turn out heads.
2. When a card is drawn at random from a pack of 52 cards and if the card is replaced, the result of the second draw is independent of the first. But if the card is not replaced then the result of the second depends on the result of the first draw.

Mathematical (priori) definition of probability : If the outcome of a trial consists n exhaustive, mutually exclusive, equally possible cases, of which m of them are favourable cases to an event E, then the **probability of the happening of the event** E, usually denoted by $P(E)$ or simply p is defined to be equal to m/n.

That is, $$P(E) = p = \frac{\text{number of favourable cases}}{\text{number of possible cases}} = \frac{m}{n}$$

The probability can atmost be equal to 1, because the number of favourable cases and the number of possible cases can atmost coincide with each other.

Since m cases are favourable to the event, it follows that $(n-m)$ cases are not favourable to the event. This set of unfavourable events is denoted by $\overline{E}$ or E'

Therefore probability of the non happening of the event usually denoted by q is given by

$$q = \frac{n-m}{n} \quad \text{or} \quad P(\overline{E}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

i.e., $$q = 1 - \frac{m}{n} = 1 - p \quad \text{or} \quad p + q = 1$$

Equivalently $P(E) + P(\overline{E}) = 1$

p is also referred to as the probability of success and q as the probability of failure. Their sum is always equal to 1.

If $P(E) = 1$, E is called a *sure event* & if $P(E) = 0$, E is called an *impossible event*.

Examples

1. *The probability of getting a 'head' in tossing of coin*

The possible outcomes are head and tail.

Number of possible (exhaustive) cases $(n) = 2$

Number of favourable cases $(m) = 1$

$\therefore$ Probability of getting head $(p) = \frac{m}{n} = \frac{1}{2}$

---

2. *The probability of getting (a) king (b) king or queen, when a card is drawn at random from a pack of 52 cards.*

Number of possible (exhaustive) cases $(n) = 52$

(a) Number of favourable cases $(m) = 4$

$\therefore$ Probability of getting a king $(p) = \frac{m}{n} = \frac{4}{52} = \frac{1}{13}$

(b) Number of favourable cases $(m) = 4 + 4 = 8$

$\therefore$ Probability of getting a king or queen $(p) = \frac{m}{n} = \frac{8}{52} = \frac{2}{13}$

---

3. *The probability of getting (a) a number greater than 2 (b) an odd number (c) an even number when a 'die' is thrown.*

Number of possible cases/outcomes (n) = 6

(a) Number of favourable outcomes = (m) = 4, since 3, 4, 5, 6 numbers are favourable to the event.

$\therefore$ Probability of getting a number greater than 2 (p) = $\frac{4}{6} = \frac{2}{3}$

(b) Number of favourable out comes (m) = 3, since 1, 3, 5 are favourable to the event.

$\therefore$ probability of getting an odd number (p) = $\frac{3}{6} = \frac{1}{2}$

(c) Similarly we have (p) = $\frac{3}{6} = \frac{1}{2}$, since 2, 4, 6 are favourable to the event.

---

4. *Probability of (a) getting a total more than 10 (b) getting a total less than 10 (c) getting a total equal to 10 when two dice are thrown simultaneously.*

Number of possible out comes when two dice are thrown simultaneously is given by (n) = 6^2 = 36

(a) Favourable outcomes are (5, 6), (6, 5) and (6, 6). That is (m) = 3

$\therefore$ the required probability (p) = $\frac{3}{36} = \frac{1}{12}$

(b) Favourable outcomes are all the outcomes except those in the first case.
That is (m) = 36 − 3 = 33

$\therefore$ the required probability (p) = $\frac{33}{36} = \frac{11}{12}$

(c) Favourable outcomes are (4, 6), (6, 4) and (5, 5). That is (m) = 3

$\therefore$ the required probability (p) = $\frac{3}{36} = \frac{1}{12}$

---

5. *Probability that a leap year will have 53 sundays.*

In a leap year there are 366 days when it comprises of 52 weeks and 2 days. These two days can be (i) Sunday & Monday (ii) Monday & Tuesday (iii) Tuesday & Wednesday (iv) Wednesday & Thursday (v) Thursday & Friday (vi) Friday and Saturday (vii) Saturday & Sunday.

Number of possible outcomes (n) = 7
Number of favourable cases (m) = 2 [(i) & (ii) outcomes]

$\therefore$ the required probability (p) = $\frac{2}{7}$

6.21 Empirical (Statistical) definition of probability

The mathematical definition of probability fails when the number of outcomes is inifinite (not exhaustive) and the outcomes are not equally likely.

The empirical definition is as follows.

If the experiment/trial is repeated a large number of times under essential identical conditions, the limiting value of the ratio of the number of times the event E happens to the total number of trials as the number of trials increases indefinitely is called the probability of the happening of the event E.

In brief we can say that, if m is the number of times in which an event E occurs in a series of n trials then the probability of happening of E is defined by,

$$p = P(E) = \lim_{n \to \infty} \frac{m}{n}$$

6.3 Probability theorems

6.31 Addition theorem of probability

The probability of the happening of one or the other mutually exclusive events is equal to the sum of the probabilities of the two events.

That is, if A, B are two mutually exclusive events then,

$$P(A \text{ or } B) = P(A) + P(B)$$

Proof : Let the total number of exhaustive, mutually exclusive and equally possible cases in the trials be n. Out of these let m_1 cases be favourable to the event A and m_2 cases be favourable to B.

Hence the number of cases favourable to either A or B is $m_1 + m_2$

$$\therefore \quad P(A \text{ or } B) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n}$$

Since m_1 cases are favourable to A, $P(A) = \frac{m_1}{n}$

Since m_2 cases are favourable to B, $P(B) = \frac{m_2}{n}$

Substituting these in the RHS of (1),

$$P(A \text{ or } B) = P(A) + P(B)$$

This proves the addition theorem of probability

Note : *The theorem is also true fo finite number of mutually exclusive events* $A_1, A_2, \ldots A_n$ *and we have,*

$$P(A_1 \text{ or } A_2 \text{ or } A_3 \ldots \text{ or } A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

An Illustration

Suppose a card is drawn at random from a pack of 52 cards and we need the probability of getting a King (K) or a Queen (Q). As there are 4 Kings and 4 Queens, 8 cards are favourable and the probability of getting any one of them is $8/52$ which is the LHS of the theorem. Now $P(K) = 4/52$ and $P(Q) = 4/52$ whose sum, as in RHS of the theorem is, $4/52 + 4/52 = 8/52$ which is same as LHS.

6.32 Product theorem of probability

If a compound event is made up of a number of independent events, the probability of the happening of the compound event is equal to the product of the probabilities of the indepenent events.

That is, If A and B are independent events then $P(A \text{ and } B) = P(A) \cdot P(B)$

Proof : Since A and B are independent events, let us suppose that the event

A occurs in a_1 ways and fails in b_1 ways ;

B occurs in a_2 ways and fails in b_2 ways, all of which are eqully likely.

$$\therefore \quad P(A) = \frac{a_1}{a_1 + b_1} \quad \text{and} \quad P(B) = \frac{a_2}{a_2 + b_2}$$

There are four possibilities for the compund event as follows.

(i)	A occurs	and B occurs in	$a_1 \cdot a_2$ ways
(ii)	A occurs	and B does not occur in	$a_1 \cdot b_2$ ways
(iii)	A does not occur	and B occursin	$b_1 \cdot a_2$ ways
(iv)	A does not occur	and B does not occur in	$b_1 \cdot b_2$ ways.

Thus the number of possible ways for the compound event is the sum of the these given by

$$a_1 a_2 + a_1 b_2 + a_2 b_1 + b_1 b_2 = (a_1 + b_1)(a_2 + b_2)$$

The number of ways favourable to the compound event is $a_1 a_2$ and hence the probability of the happening of the compound event is

$$P(A \text{ and } B) = \frac{a_1 a_2}{(a_1 + b_1)(a_2 + b_2)}$$

$$= \frac{a_1}{(a_1 + b_1)} \cdot \frac{a_2}{(a_2 + b_2)} = P(A) \cdot P(B)$$

Thus $\mathbf{P(A \text{ and } B) = P(A) \cdot P(B)}$.

This proves the product theorem of probability.

Note : *The theorem is also true for finite number of independent events*

$A_1, A_2, A_3, \ldots$ *and we have,*

$$P(A_1 \text{ and } A_2 \ldots \text{ and } A_n) = P(A_1) \cdot P(A_2) \ldots P(A_n)$$

An Illustration

Suppose that there is a function for which two guests G_1 and G_2 are expected to be present. The four possibilities are that (i) both G_1 and G_2 are present (ii) G_1 is present and G_2 is absent (iii) G_1 is absent and G_2 is present (iv) both are absent. These are equally likely. The only case fabourable is the presence of both G_1 and G_2. The corresponding probability is 1/4 which is LHS of the theorem.

Looking independently, the probability of G_1 being present is 1/2 because G_1 present or G_1 absent are the only two possibalities. Similarly the probability of G_2 being present is also 1/2 and the product of these two probabilities as in RHS of the theorem is $1/2 \cdot 1/2 = 1/4$ which is same as LHS.

WORKED PROBLEMS

1. *A box contains 3 white, 5 black and 6 red balls. If a ball is drawn at random what is the probability that it is either red or white ?*

>> Total number of balls in the box = 14

$$P\text{ (getting a red ball)} = \frac{6}{14} \; ; \; P\text{ (getting a white ball)} = \frac{3}{14}$$

$$\therefore \quad P\text{ (getting red or white)} = \frac{6}{14} + \frac{3}{14} = \frac{9}{14}$$

(*By addition theorem since the two events are exclusive*)

Thus the required probability is 9/14 .

---

2. *An urn contains 2 white and 2 black balls and a second urn contains 2 white and 4 black balls. If one ball is drawn at random from each urn what is the probability that they are of the same colour ?*

>> Total number of balls in the first urn $= 2W + 2B = 4$

Total number of balls in the second urn $= 2W + 4B = 6$

Case-(i) : Suppose both the balls drawn are white, then the probability is $\frac{2}{4} \times \frac{2}{6} = \frac{1}{6}$

Case-(ii) : Supose both the balls drawn are black, then the probability is $\frac{2}{4} \times \frac{4}{6} = \frac{1}{3}$

Since either of these two cases are favourable to the event, the required probability by addition theorem is $1/6 + 1/3 = 1/2$

Thus the required probability is $1/2$.

---

3. *An urn A contains 2 white and 4 black balls. A second urn contains 5 white and 7 black balls. A ball is transferred from A to B and then a ball is drawn from B. Find the probability that it is white.*

>> Total number of balls in urn $A = 2W + 4B = 6$

Total number of balls in urn $B = 5W + 7B = 12$

Since a ball is transferred from A to B, two cases arise.

Case-(i) Suppose the transferred ball is white.

Probability of the transfer of a white ball is $2/6 = 1/3$

Then urn B will have $6W$ and $7B = 13$ balls.

Hence probability of getting a white ball from B after the transfer is $6/13$.

$\therefore$ Probability of transferring a white ball and getting white from B is

$$\frac{1}{3} \times \frac{6}{13} = \frac{2}{13}$$

Case-(ii) Suppose the transferred ball is black. Probability of transfer is $4/6 = 2/3$

Then urn B will have $5W$ and $8B = 13$ balls.

Hence the probability of getting a white ball after the transfer is $5/13$.

$\therefore$ Probability of transferring a black ball and getting white from B is

$$\frac{2}{3} \times \frac{5}{13} = \frac{10}{39}$$

Either of these two cases are favourable to the event.

Thus the required probability by addition theorem is $\mathbf{\frac{2}{13} + \frac{10}{39} = \frac{6 + 10}{39} = \frac{16}{39}}$

---

4. *A bag contains 4 white and 2 black balls. Another bag contains 3 white and 5 black balls. If a ball is drawn from each, find the probability that (i) both are white (ii) both are black (iii) one is black and another is white.*

>> Total number of balls in the first bag (B_1) is $4W + 2B = 6$ balls.

Total number of balls in the second bag (B_2) is $3W + 5B = 8$ balls

(i) Probability of getting both white

$P(W \text{ and } W)$ is $\frac{4}{6} \times \frac{3}{8} = \frac{1}{4}$ by applying product theorem.

(ii) Probability of getting both black

$P(B \text{ and } B)$ is $\frac{2}{6} \times \frac{5}{8} = \frac{5}{24}$

(iii) $P(B \text{ from } B_1 \text{ and } W \text{ from } B_2) = \frac{2}{6} \times \frac{3}{8} = \frac{1}{8}$

or $P(W \text{ from } B_1 \text{ and } B \text{ from } B_2) = \frac{4}{6} \times \frac{5}{8} = \frac{5}{12}$

Either of these two cases are favourable to the event.

Thus the required probability by addition theorem is $\frac{1}{8} + \frac{5}{12} = \frac{13}{24}$

---

5. *Two cards are drawn in succession from a pack of 52 cards. Find the probability that the first is king and the second is queen if the first card is (a) replaced (b) not replaced.*

>> (a) Probability of getting king $= \frac{4}{52} = \frac{1}{13}$

Probability of getting queen $= \frac{4}{52} = \frac{1}{13}$

Thus P (getting king and queen) $= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$ (*When the card is replaced*)

(b) In the second case as before, the probability of getting king is 1/13.

If the card is not replaced then there will be $52 - 1 = 51$ cards in the pack and the probability of getting queen is 4/51.

Thus, P (getting king and queen) $= \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$ (*When the card is not replaced*)

---

6. *5 balls are drawn at random from a bag of 6 white and 4 black balls. What is the chance that 3 of them are white and 2 are black ?*

>> Total number of balls $= 6W + 4B = 10$ balls.

The number of possible ways of selecting 5 balls out of 10 is $^{10}C_5$

$3W$ from $6W$ can be selected in $^6C_3$ ways & $2B$ from $4B$ can be selected in $^4C_2$ ways.

$\therefore$ $3W$ and $2B$ can be selected in $^6C_3 \times {}^4C_2$ ways.

Thus the required probability of getting 3*W* and 2 *B* out of 10 balls is

$$\frac{{}^{6}C_3 \times {}^{4}C_2}{{}^{10}C_5} = \frac{10}{21} \quad (\text{On simplification})$$

---

7. *There are 10 students of which three are graduates. If a committee of five is to be formed, what is the probability that there are (i) only 2 graduates (ii) atleast 2 graduates.*

>> As there are 3 graduates, the other 7 be taken as non-graduates. Since a committee of 5 is to be formed, there are ${}^{10}C_5$ ways of selecting 5 people out of 10.

(i) Only 2 graduates imply the other 3 must be nongraduates.

The number of ways of selecting 2 out of 3 and 3 out of 7 is ${}^{3}C_2 \times {}^{7}C_3$.

Thus the required probability of having only 2 graduates is

$$\frac{{}^{3}C_2 \times {}^{7}C_3}{{}^{10}C_5} = \frac{5}{12} \quad (\text{On simplification})$$

(ii) At least 2 graduates, will give rise to the following two cases.

2 graduates + 3 nongraduates or 3 graduates + 2 nongraduates.

The probability in the first case as already computed is $5/12$.

The probability of having 3 graduates and 2 nongraduates is given by

$$\frac{{}^{3}C_3 \times {}^{7}C_2}{{}^{10}C_5} = \frac{1}{12}$$

Either of these two cases is favourable to the event.

Thus the required probability by addition theorem is $\frac{5}{12} + \frac{1}{12} = \frac{1}{2}$

---

8. *The probability that a person A solves the problem is 1/3, that of B is 1/2 and that of C is 3/5. If the probelm is simultaneously assigned to all of them what is the probability that the problem is solved ?*

>> It should be noted that even if any one of them solves the problem, it is presumed that the problem is solved.

Hence we shall consider the probabilities for the following cases.

A not solving the problem $= 1 - \frac{1}{3} = \frac{2}{3}$

B not solving the problem $= 1 - \frac{1}{2} = \frac{1}{2}$

C not solving the problem $= 1 - \frac{3}{5} = \frac{2}{5}$

$\therefore$ the probability that the problem is not solved at all is $\frac{2}{3} \times \frac{1}{2} \times \frac{2}{5} = \frac{2}{15}$

Equivalently we can say that $q = \frac{2}{15}$

But $p + q = 1$ where p isthe probability of solving the problem and q is the probability of not solving the problem.

The required probability is $p = 1 - q$

Thus $p = 1 - \frac{2}{15} = \frac{13}{15}$

---

9. *If two numbers are selected from the set of numbers* $\{0, 1, 2, 3, \ldots 9\}$ *find the chance that their sum is equal to 10.*

>> The set has 10 numbers and the number of ways of selecting 2 out of 10 is ${}^{10}C_2$

The various possible pairs of numbers giving 10 as their sum are

$(1, 9)$; $(2, 8)$; $(3, 7)$; $(4, 6)$, being 4 possibilities.

Thus the required probability is $\frac{4}{{}^{10}C_2} = \frac{4}{45}$

---

10. *From 6 positive and 8 negative numbers, 4 numbers are chosen at random (without replacement) and multiplied. What is the probability that the product is a positive number.*

>> To get the product of 4 numbers positive we have the following 3 possibilities where P denotes a positive number and N denotes a negative number.

$4P$ and $0N$; $2P$ and $2N$; $0P$ and $4N$

$\therefore$ the associated probability is given by

$$\frac{^6C_4 \times {^8C_0}}{^{14}C_4} + \frac{^6C_2 \times {^8C_2}}{^{14}C_4} + \frac{^6C_0 \times {^8C_4}}{^{14}C_4} = \frac{1}{1001}(15 + 420 + 70) = 0.5045$$

Thus the required probability is 0.5045

---

11. *A box contains tags marked 1, 2, ... n. Two tags are chosen at random. Find the probability that numbers on the tags will be consecutive integers if (i) tags are chosen with replacement (ii) tags are chosen without replacement.*

>> (i) No. of tags $= n$ and selection of 2 tags can have the pairs of numbers

$$\left.\begin{matrix}(1, 2), (1, 3), (1, 4), \cdots (1, n) \\ (2, 1), (2, 3), (2, 4), \cdots (2, n) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (n, 1), (n, 2), (n, 3), \cdots (n, n-1)\end{matrix}\right\} = n(n-1) \text{ ways}$$

On replacement, the total number of ways $= n + n(n-1) = n^2$

The favourable cases are,

$$\left\{\begin{matrix}(1, 2), (2, 3), (3, 4), \cdots (n-1, n) \\ (2, 1), (3, 2), (4, 3), \cdots (n, n-1)\end{matrix}\right\} = (n-1) + (n-1) = 2(n-1)$$

Thus the required probability $= \dfrac{2(n-1)}{n^2}$

(ii) 2 tags can be selected out of n tags in $^nC_2$ ways. The favourable cases are

$$(1, 2), (2, 3), \cdots (n-1, n) = (n-1) \text{ ways}$$

Thus the required probability $= \dfrac{n-1}{^nC_2} = \dfrac{(n-1)}{n(n-1)/2} = \dfrac{2}{n}$

---

12. *A bag contains 10 white and 3 red balls while another bag contains 3 white and 5 red balls. 2 balls are drawn at random from the first bag and put in the second bag. Then a ball is drawn at random from the second bag. What is the probability that it is a white ball ?*

>> $B_1 : 10W + 3R = 13$ balls

$B_2 : 3W + 5R = 8$ balls

When two balls are drawn at random from the first bag, the number of possible ways is $^{13}C_2$. The outcomes may be

$(W \& W), (R \& R), (W \& R)$

$$\therefore \quad P(2W) = \frac{^{10}C_2}{^{13}C_2} \; ; \; P(2R) = \frac{^{3}C_2}{^{13}C_2} \; ; \; P(1W \text{ and } 1R) = \frac{3 \times 10}{^{13}C_2}$$

$B_2 + 2W \quad = 5W + 5R = 10$ balls ; $P(W) = 5/10$

$B_2 + 2R \quad = 3W + 7R = 10$ balls ; $P(W) = 3/10$

$B_2 + (1W + 1R) = 4W + 6R = 10$ balls ; $P(W) = 4/10$

Probability in each of these cases are respectively

$$\frac{^{10}C_2}{^{13}C_2} \times \frac{5}{10} \; ; \; \frac{^{3}C_2}{^{13}C_2} \times \frac{3}{10} \; ; \; \frac{30}{^{13}C_2} \times \frac{4}{10}$$

Since either of these 3 cases are favourable to the event the required probability is the sum of all these.

Thus the probability is given by

$$\frac{1}{10 \cdot {^{13}C_2}} [\, ^{10}C_2 \cdot 5 + {^{3}C_2} \cdot 3 + 30 \cdot 4 \,] = \mathbf{0.454}$$

---

13. *Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1 girl and 3 boys. One child is selected at random from each group. Find the probability of selecting 1 girl and 2 boys.*

>> The required event of selecting 1 girl (*G*) and 2 boys (*B*) can be in the following ways which are mutually exclusive. The associated probabilities is also worked out.

Group	1	2	3	Probabilities
Option-1	*G*	*B*	*B*	3/4. 2/4. 3/4 = 9/32
Option-2	*B*	*G*	*B*	1/4. 2/4. 3/4 = 3/32
Option-3	*B*	*B*	*G*	1/4. 2/4. 1/4 = 1/32

The required probability is the sum of all these probabilities.

Thus we have, 9/32 + 3/32 + 1/32 = **13/32**

---

14. *A bag contains 40 tickets numbered 1, 2, 3, ⋯40, of which four are drawn at randam and arranged in the ascending order.* $(t_1 < t_2 < t_3 < t_4)$
Find the probability of t_3 *being 25.*

>> 4 tickets out of 40 can be selected in 40_{C_4} ways.

If $t_3 = 25$ then t_1 and t_2 must be from 1 to 24 which can be selected in 24_{C_2} ways.

Also t_4 must be from the remaining 15 tickets numbered 26 to 40 and there are 15 ways of selecting t_4

Hence the number of ways favourable to the event is $24_{C_2} \times 15 = 4140$

$\therefore$ probability of t_3 being $25 = \dfrac{4140}{40_{C_4}} = \dfrac{4140}{91390}$

Thus the probability of t_3 being 25 is 0.0453

---

15. *Two dice are thrown. Find the proability of (a) getting an odd number on the one and a multiple of 3 on the other (b) one of the dice showed 3 and sum on the two dice is 9 (c) sum on the two dice is 9 (d) Sum on the two dice is 13.*

>> Here we need to list out all the various possible outcomes ($6^2 = 36$) of the simultaneous throwing of two dice. They are as follows.

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Total number of outcomes = 36

(a) The favourable outcomes in the case of getting an odd number on one 'die' and a multiple of 3 on the other 'die' is as follows.

(i) (1, 3) (1, 6) (3, 3) (3, 6) (5, 3) (5, 6)

(ii) (3, 1), (6, 1), (6, 3), (3, 5), (6, 5)

Total number of favourable outcomes = 11

Thus the required probability = 11/36

(b) The favourable outcomes in the case of one 'die' showing 3 and sum of the two dice is 9 are (3 , 6) and (6 , 3) being equal to 2.

Thus the required probability is 2/36 = 1/18

(c) The favourable outcomes in the case of sum on the two dice is 9 are (3 , 6) (4 , 5) (5 , 4) (6 , 3) being equal to 4.

Thus the required probability is 4/36 = 1/9

(d) Sum on the two dice equal to 13 is an impossible outcome.

Thus the required probability = 0

6.4 Probability associated with set theory

6.41 Recapitulation of Set Theory

A *set* is a collection of objects and the objects are called members or elements of the set. Sets are usually denoted by $A, B, C, \ldots$

If an element x belongs to A we write $x \in A$, otherwise we write $x \notin A$.

If B is a set such that every element of B also belongs to A we say that B is a ***subset*** of A written as $B \subset A$. This is equivalent to writing $A \supset B$ read as A contains B. Every set is a sub set of itself.

$$A \subset B \text{ and } B \subset A \text{ implies } A = B.$$

If $A \subset B$ and $A \neq B$ we say that A is a ***proper sub set*** of B.

All sets under consideration are assumed to be sub sets of some fixed set called the ***universal set*** usually denoted by U.

A set which contains no elements is called a ***null set*** or ***empty set*** usually denoted by ϕ. For any set A, $\phi \subset A \subset U$.

A universal set U is represented geometrically by a set of points inside a rectangle. Sub sets of U are represented by circles inside the rectangle. Such a geometrical representation is called a *Venn diagram.*

Set Operations

The set of all elements which belong to either A or B or both is called the ***union*** of two sets A and B denoted by $A \cup B$.

The set of all elements which belong to both A and B is called the ***intersection*** of the sets A and B denoted by $A \cap B$.

The set of all elements which belong to A but does not belong to B is called the ***difference*** of A and B denoted by $A - B$.

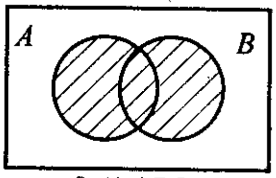
The set of elements of U which does not belong to A is called the ***compliment*** of A denoted by A'.

$A' = U - A$. Notations $\overline{A}$, A^c are also used for the complement of A.

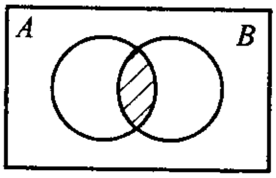
Two sets A and B are said to be ***disjoint*** if $A \cap B = \phi$

The set of all subsets of A is called the ***power set*** of A. It may be noted that $\phi \subset A$, $A \subset A$.

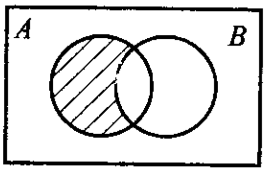
Venn diagrams for set operations



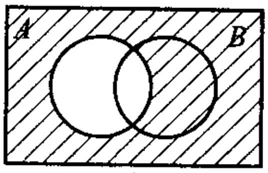
$[A \cup B]$



$[A \cap B]$



$[A - B]$



$[A']$

(Shaded portions in each diagram indicate the set operation)

Some important laws on set operations

1. $A \cup B = B \cup A$; $A \cap B = B \cap A$ [*Commutative laws*]
2. $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$ [*Associative laws*]
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [*Distributive laws*]
4. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$; $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ [*De Morgan's laws*]
5. $A - B = A \cap \overline{B}$; $\overline{(\overline{A})} = A$

6.42 Random experiments, Sample space and Events

The experiments (*trials*) when performed repeatedly giving different results (*out comes*) are called ***random experiments.***

A set S consisting of all possible out comes of a random experiment is called a ***sample space,*** which corresponds to the *universal set.*

If a sample space has finite number of elements, then it is called a finite sample space otherwise it is called an infinite sample space.

An ***event*** E is a ***sub set*** of the sample space S. A particular out come that is an element in S is called a ***sample.***

If E_1 and E_2 are two events then

(i) $E_1 \cup E_2$ is the event E_1 or E_2 or both.

(ii) $E_1 \cap E_2$ is the event E_1 and also E_2

(iii) E_1' is the event that occurs if E_1 does not occur.

(iv) $E_1 - E_2$ is the event E_1 but not E_2

Example - 1 In throwing a cubical 'die' a number appears at the top. The sample space consist of six possible numbers.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Suppose E_1 is the event of getting an odd number at the top and E_2 is the event of getting an even number then

$$E_1 = \{1, 3, 5\} ; E_2 = \{2, 4, 6\}.$$ E_1 and E_2 are sub sets of S.

Example - 2 Suppose a coin is tossed twice and E is the event of getting atleast one head, then we have

$$S = \{HH, HT, TH, TT\} , E = \{HH, HT, TH\}$$

6.43 Axioms of Probability *(Axiomatic definition of probability)*

If S is the sample space and E is the set of all events then to each event A in E we associate a unique real number $P = P(A)$ known as the **probability of the event** A, if the following axioms are satisfied.

These are known as the ***axioms of probability***.

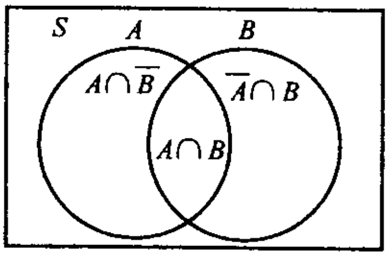
1. $P(S) = 1$
2. For every event A in E $0 \leq P(A) \leq 1$
3. If $A_1, A_2, A_3, \ldots A_n$ are mutually exclusive events of E then
$P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$

6.44 Addition rule

If A and B are any two events of S which are not mutually exclusive then

$$P(A \cup B) = P(A)+P(B)-P(A \cap B)$$

Proof : We prove the result using the following Venn diagram.



From the figure

$$A = (A \cap \overline{B}) \cup (A \cap B)$$
$$B = (\overline{A} \cap B) \cup (A \cap B)$$

$\Rightarrow$ $P(A) = P(A \cap \overline{B})+P(A \cap B)$, since $A \cap \overline{B}$ and $A \cap B$ are disjoint.

$P(B) = P(\overline{A} \cap B)+P(A \cap B)$, since $\overline{A} \cap B$ and $A \cap B$ are disjoint.

Here we have used the third axiom of probability that

$P(A_1 \cup A_2) = P(A_1)+P(A_2)$ if A_1 and A_2 are mutually exclusive.

Now $P(A)+P(B) = [P(A \cap \overline{B})+P(A \cap B)+P(\overline{A} \cap B)]+P(A \cap B)$

That is $P(A)+P(B) = [P(A \cup B)]+P(A \cap B)$

Thus $\mathbf{P(A \cup B) = P(A)+P(B)-P(A \cap B)}$

Remarks :

1. *If A and B are mutually exclusive then* $P(A \cap B) = 0$

$\therefore$ $P(A \cup B) = P(A)+P(B)$

This is same as the addition theorem proved earlier **(6.31)**

2. *Since* $P(A \cap B) \geq 0$, $P(A \cup B) \leq P(A)+P(B)$

6.45 Conditional probability

In many situations the probabilities of two or more events depend on one another. In otherwords, the happening of one event depends on the happening of the other.

Definition : Let A and B be the two events. Probability of the happening of the event B when the event A has already happened is called the ***conditional probability*** denoted by $P(B/A)$.

That is to say that, $P(B/A)$ is the probability of B given A.

$$P(B/A) = \frac{\text{Probability of the occurence of both } B \text{ and } A}{\text{Probability of the occurence of the given event } A}$$

That is $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$ ... (1)

Also $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ ... (2)

6.46 Multiplication rule

We have from (1), $P(A \cap B) = P(A) \cdot P(B/A)$ ... (3)

where $P(A) > 0$. This is called the ***multiplication rule of probability.***

Remark : *If A and B are two independent events then $P(B/A) = P(B)$. Hence (3) becomes*

$$P(A \cap B) = P(A) \cdot P(B)$$

That is, $P(A \cap B) = P(A) \cdot P(B) \Leftrightarrow A$ *and* B *are independent.*

This is same as a product theorem proved earlier **(6.32)**

Illustrative Example

Suppose a cubical 'die' is thrown. The sample space $S = \{1, 2, 3, 4, 5, 6\}$

Let A be the event of getting an odd number. Hence

$$A = \{1, 3, 5\}$$

In the next trial let B be the event of getting a number less than 4

Hence, $B = \{1, 2, 3\}$

Now $P(B/A)$ is the probability of getting a number less than 4 being odd number.

That is, $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

Here, $A \cap B = \{1, 3\}$ and $P(A \cap B) = \dfrac{2}{6} = \dfrac{1}{3}$; $P(A) = \dfrac{3}{6} = \dfrac{1}{2}$

Hence, $P(B/A) = \dfrac{1/3}{1/2} = \dfrac{2}{3}$

6.47 Baye's theorem on conditional probability

Let $A_1, A_2, \cdots A_n$ be a set of exhaustive and mutually exclusive events of the sample space S with $P(A_i) \neq 0$ for each i. If A is any other event associated with A_i, $(A \subset \bigcup_{i=1}^{n} A_i)$ with $P(A) \neq 0$ then

$$P(A_i/A) = \frac{P(A_i)P(A/A_i)}{\sum_{i=1}^{n} P(A_i)P(A/A_i)}$$

Proof : We have $S = A_1 \cup A_2 \cup \cdots A_n$ and $A \subset S$

$\therefore \quad A = S \cap A = (A_1 \cup A_2 \cup \cdots \cup A_n) \cap A$

Using distributive law in the RHS we have

$$A = (A_1 \cap A) \cup (A_2 \cap A) \cup \cdots (A_n \cap A)$$

Since $A_i \cap A$ for $i = 1$ to n are mutually exclusive, we have by applying the addition rule of probability,

$$P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \cdots + P(A_n \cap A)$$

Now applying multiplication rule onto each term in the RHS we have,

$$P(A) = P(A_1)P(A/A_1) + P(A_2)P(A/A_2) + \cdots + P(A_n)P(A/A_n)$$

That is, $\quad P(A) = \sum_{i=1}^{n} P(A_i)P(A/A_i)$

The conditional probability of A_i for any i given A, is defined by

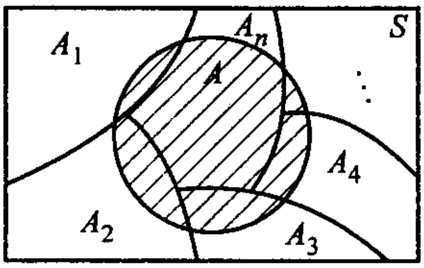
$$P(A_i/A) = \frac{P(A_i \cap A)}{P(A)} = \frac{P(A_i)P(A/A_i)}{P(A)}$$

Using (1) in the denominator of the RHS we have,

$$P(A_i/A) = \frac{P(A_i)P(A/A_i)}{\sum_{i=1}^{n} P(A_i)P(A/A_i)}$$

This proves Baye's theorem for conditional probability.

Note: *The mutually exclusive events $A_1, A_2, \cdots A_n$ of the sample space S form a partition of S and the event A is shaded in the following figure.*



In this context Baye's theorem can also be stated as follows.

Suppose $A_1, A_2, \cdots A_n$ form a partition of the sample space S & A is any other event, then

$$P(A_i/A) = \frac{P(A_i)P(A/A_i)}{\sum_{i=1}^{n} P(A_i)P(A/A_i)}$$

WORKED PROBLEMS

16. *Prove the following :*

(i) $P(\phi) = 0$

(ii) $P(\overline{A}) = 1 - P(A)$ *where $\overline{A}$ is the compliment of A.*

>> **(i)** We have $A \cup \phi = A$ for any set A.

$\Rightarrow \quad P(A \cup \phi) = P(A)$

That is, $\quad P(A) + P(\phi) = P(A)$ by the probability axiom (3)

or $\quad P(\phi) = P(A) - P(A) = 0$

Thus $\mathbf{P(\phi) = 0}$

(ii) The events $A, \overline{A}$ are disjoint and their union is the sample space S.

That is, $\quad A \cup \overline{A} = S$

$\Rightarrow \quad P(A \cup \overline{A}) = P(S)$ or $P(A) + P(\overline{A}) = 1$

Hence we have used probability axioms (3) and (1).

Thus $\mathbf{P(\overline{A}) = 1 - P(A)}$

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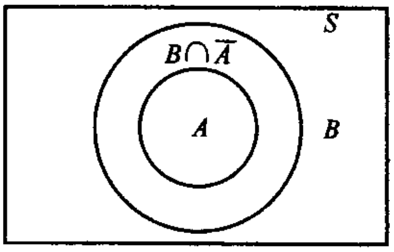
17. *If $A \subset B$ then prove that $P(A) \le P(B)$*

>>



Since $A \subset B$, $\overline{A} \cap B$ and A are mutually exclusive and their union is B.

That is $B = (\overline{A} \cap B) \cup A$

$\Rightarrow$ $P(B) = P(\overline{A} \cap B) + P(A)$, by using probability axiom (3) in the RHS.

or $P(B) - P(A) = P(\overline{A} \cap B)$

That is $P(B) - P(A) \ge 0$, by using axiom (2).

Thus $\mathbf{P(B) \ge P(A)}$ or $\mathbf{P(A) \le P(B)}$

---

18. *If A and B are independent events, show that the events*

(i) $\overline{A}$ and $\overline{B}$ (ii) $\overline{A}$ and B (iii) A and $\overline{B}$ are also independent.

>> (i) We have by De - Morgan's law of sets,

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Now $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$

$= 1 - P(A \cup B)$, since $P(\overline{X}) = 1 - P(X)$ for any set X.

$= 1 - [P(A) + P(B) - P(A \cap B)]$, by addition theorem.

$= 1 - [P(A) + P(B) - P(A) \cdot P(B)]$

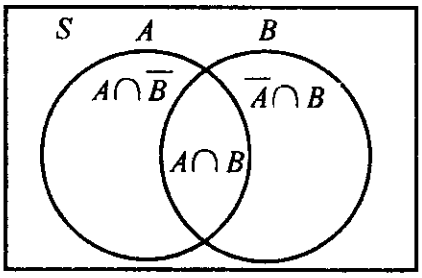
Here we have used $P(A \cap B) = P(A) \cdot P(B)$ as A & B are independent events.

Hence $P(\overline{A} \cap \overline{B}) = [1 - P(A)][1 - P(B)]$ by factorizing the RHS.

That is, $P(\overline{A} \cap \overline{B}) = P(\overline{A}) \cdot P(\overline{B})$

Thus $\overline{A}$ **and** $\overline{B}$ **are independent.**

(ii)



We have from the figure, $B = (A \cap B) \cup (\overline{A} \cap B)$

Since $A \cap B$ and $\overline{A} \cap B$ are disjoint we have

$$P(B) = P(A \cap B) + P(\overline{A} \cap B)$$

or $$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A) \cdot P(B)$$

$$= P(B)[1 - P(A)] = P(B) \cdot P(\overline{A})$$

$\therefore$ $$P(\overline{A} \cap B) = P(\overline{A}) \cdot P(B)$$

Thus, **$\overline{A}$ and B are independent.**

(iii) We have from the figure,

$$A = (A \cap \overline{B}) \cup (A \cap B)$$

$\therefore$ $$P(A) = P(A \cap \overline{B}) + P(A \cap B)$$

or $$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A) \cdot P(B)$$

$$= P(A)[1 - P(B)] = P(A)P(\overline{B})$$

$\therefore$ $$P(A \cap \overline{B}) = P(A) \cdot P(\overline{B})$$

Thus **A and $\overline{B}$ are independent.**

---

19. *If A, B, C are mutually independent events, show that the events $A \cup B$ and C are also independent.*

>> We shall prove that $P[(A \cup B) \cap C] = P(A \cup B) \cdot P(C)$

Now $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ by distributive law.

$\Rightarrow$ $$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$$

$$= P(A \cap C) + P(B \cap C) - P\{(A \cap C) \cap (B \cap C)\}$$

We have used addition rule in the RHS.

Further we have,

$$P[(A \cup B) \cap C] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

$$= P(A) \cdot P(C) + P(B) \cdot P(C) - P(A) \cdot P(B) \cdot P(C)$$

since A, B, C are mutually independent.

That is, $$P[(A \cup B) \cap C] = P(C)[P(A) + P(B) - P(A) \cdot P(B)]$$

$$= P(C)[P(A) + P(B) - P(A \cap B)]$$

$$= P(C) \cdot P(A \cup B)$$

Hence $P[(A \cup B) \cap C] = P(A \cup B) \cdot P(C)$

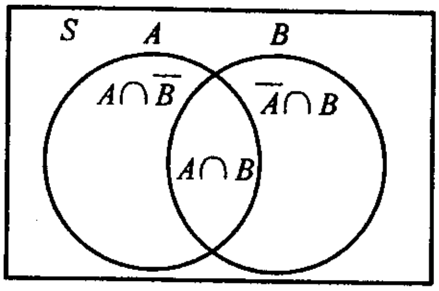
Thus we conclude that $A \cup B$ **and** C **are independent.**

---

20. *If A and B are two events in a sample space S, show that*

1. $P[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = P(A) + P(B) - 2P(A \cap B)$
2. $P[\overline{A} \cap \overline{B}] = 1 + P(A \cap B) - P(A) - P(B)$

>>



1. We have from the figure,

$$A = (A \cap \overline{B}) \cup (A \cap B) \text{ and } B = (A \cap B) \cup (\overline{A} \cap B)$$

Since $A \cap \overline{B}$, $A \cap B$, $\overline{A} \cap B$ are mutually disjoint we have by addition rule

$$P(A) = P(A \cap \overline{B}) + P(A \cap B)$$

and $P(B) = P(A \cap B) + P(\overline{A} \cap B)$

Adding these we get,

$$P(A) + P(B) = P(A \cap \overline{B}) + P(\overline{A} \cap B) + 2P(A \cap B)$$

$\therefore \quad P(A \cap \overline{B}) + P(\overline{A} \cap B) = P(A) + P(B) - 2P(A \cap B)$

Since $A \cap \overline{B}$ and $\overline{A} \cap B$ are disjoint the result can be put in the form,

$$\mathbf{P[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = P(A) + P(B) - 2P(A \cap B)}$$

2. $\overline{A \cup B} = \overline{A} \cap \overline{B}$ by De-Morgan's law.

$\Rightarrow \quad P(\overline{A \cup B}) = P(\overline{A} \cap \overline{B})$

i.e., $\quad 1 - P(A \cup B) = P(\overline{A} \cap \overline{B})$

i.e., $\quad 1 - \{P(A) + P(B) - P(A \cap B)\} = P(\overline{A} \cap \overline{B})$

Thus $\quad \mathbf{P(\overline{A} \cap \overline{B}) = 1 + P(A \cap B) - P(A) - P(B)}$

---

21. *If A and B are events with $P(A \cup B) = 7/8$, $P(A \cap B) = 1/4$ and $P(\overline{A}) = 5/8$ find $P(A)$, $P(B)$ and $P(A \cap \overline{B})$*

>> $P(A) = 1 - P(\overline{A}) = 1 - (5/8) \quad \therefore \mathbf{P(A) = 3/8}$

We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

i.e., $7/8 = 3/8 + P(B) - 1/4$

$\therefore$ $P(B) = 7/8 - 3/8 + 1/4$ or $\mathbf{P(B) = 3/4}$

Also we have $P(A \cap \overline{B}) = P(A) - P(A \cap B) = 3/8 - 1/4 = 1/8$

$\therefore$ $\mathbf{P(A \cap \overline{B}) = 1/8}$

---

22. *If A and B are events with* $P(A) = 1/2$, $P(A \cup B) = 3/4$ *and* $P(\overline{B}) = 5/8$, *find the following.* $P(A \cap B)$, $P(\overline{A} \cap \overline{B})$, $P(\overline{A} \cup \overline{B})$, $P(\overline{A} \cap B)$

>> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

i.e., $3/4 = 1/2 + [1 - (5/8)] - P(A \cap B)$

$\therefore$ $\mathbf{P(A \cap B) = 1/8}$

Now $\overline{A \cup B} = \overline{A} \cap \overline{B}$ *(De - Morgan's law)*

$\Rightarrow$ $P(\overline{A \cup B}) = P(\overline{A} \cap \overline{B})$

i.e., $1 - P(A \cup B) = P(\overline{A} \cap \overline{B})$

i.e., $1 - (3/4) = P(\overline{A} \cap \overline{B})$

$\therefore$ $\mathbf{P(\overline{A} \cap \overline{B}) = 1/4}$

Next, $\overline{A \cap B} = \overline{A} \cup \overline{B}$ *(De - Morgan's law)*

$P(\overline{A \cap B}) = P(\overline{A} \cup \overline{B})$

i.e., $1 - P(A \cap B) = P(\overline{A} \cup \overline{B})$

i.e., $1 - (1/8) = P(\overline{A} \cup \overline{B})$

$\therefore$ $\mathbf{P(\overline{A} \cup \overline{B}) = 7/8}$

Also, $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

$= \{1 - P(\overline{B})\} - P(A \cap B) = 1 - (5/8) - (1/8) = 1/4$

$\therefore$ $\mathbf{P(\overline{A} \cap B) = 1/4}$

---

23. *If A and B are events with* $P(A \cup B) = 3/4$; $P(\overline{A}) = 2/3$ *and* $P(A \cap B) = 1/4$, *find* $P(A)$, $P(B)$ *and* $P(A \cap \overline{B})$

>> $P(A) = 1 - P(\overline{A}) = 1 - (2/3)$ $\therefore$ $\mathbf{P(A) = 1/3}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

i.e., $3/4 = 1/3 + P(B) - 1/4$

or $\quad P(B) = 3/4 - 1/3 + 1/4 \qquad \therefore \; \mathbf{P(B) = 2/3}$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= 1/3 - 1/4 \qquad \therefore \; \mathbf{P(A \cap \bar{B}) = 1/12}$$

---

24. *Prove that*

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) + P(A \cap B \cap C) - P(A \cap B) - P(B \cap C) - P(C \cap A)$$

>> Let us write $A \cup B \cup C$ as $A \cup (B \cup C)$ and apply the addition rule.

i.e., $\quad P(A \cup B \cup C) = P(A) + P(B \cup C) - P[A \cap (B \cup C)]$

i.e., $\quad P(A \cup B \cup C) = P(A) + P(B \cup C) - P[(A \cap B) \cup (A \cap C)]$

by using distributive law for the third term in the RHS.

Applying the addition rule for the second and third terms in the RHS we have,

$$P(A \cup B \cup C) = P(A) + [P(B) + P(C) - P(B \cap C)] - \{P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]\}$$

But $\quad (A \cap B) \cap (A \cap C)$ is $(A \cap B \cap C)$

$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

Thus we have proved that

$$\mathbf{P(A \cup B \cup C) = P(A) + P(B) + P(C) + P(A \cap B \cap C) - P(A \cap B) - P(B \cap C) - P(C \cap A).}$$

---

25. *If A_1 and A_2 are two events with probabilities* 0.25 *and* 0.5 *corresponding to A_1 and $(A_1 \cup A_2)$ respectively, find the probability of A_2 if*

1. *A_1 and A_2 are mutually exclusive.*
2. *A_1 and A_2 are independent.* **3.** *A_2 contains A_1.*

>> By data $P(A_1) = 0.25$ and $P(A_1 \cup A_2) = 0.5$

1. We have $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$ ...(1)

If A_1 and A_2 are mutually exclusive, $A_1 \cap A_2 = \phi$

$\therefore \quad P(A_1 \cap A_2) = P(\phi) = 0$

Hence (1) becomes,

$$0.5 = 0.25 + P(A_2) - 0 \quad \therefore \quad \mathbf{P(A_2) = 0.25}$$

2. If A_1 and A_2 are independent

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

Now we have from (1),

$$0.5 = 0.25 + P(A_2) - 0.25 \cdot P(A_2)$$

or $\quad 0.75\, P(A_2) = 0.25 \quad \therefore \quad \mathbf{P(A_2) = 1/3}$

3. A_2 contains $A_1 \Rightarrow A_1 \subset A_2 \Rightarrow A_1 \cup A_2 = A_2$

Hence $P(A_1 \cup A_2) = P(A_2) \quad \therefore \quad \mathbf{P(A_2) = 0.5}$

---

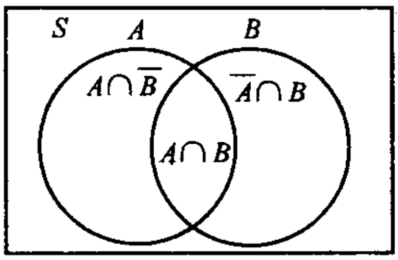
26. *If A and B are two events prove the following results.*

1. $P(A/\overline{B}) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$ *where* $P(B) \neq 1$

and hence deduce that $P(A \cap B) \geq P(A) + P(B) - 1$

2. $P(\overline{A}/B) = 1 - \dfrac{P(A \cap B)}{P(B)}$

>>



1. $\quad P(A/\overline{B}) = \dfrac{P(A \cap \overline{B})}{P(\overline{B})}$ ...(1)

From the figure, $A = (A \cap \overline{B}) \cup (A \cap B)$

$\Rightarrow \quad P(A) = P(A \cap \overline{B}) + P(A \cap B)$

$\therefore \quad P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Also we have $P(\overline{B}) = 1 - P(B)$

Using these in (1) we have,

$$\mathbf{P(A/\overline{B}) = \frac{P(A) - P(A \cap B)}{1 - P(B)}} \quad \text{where } P(B) \neq 1$$

Further $P(A/\overline{B}) \le 1$ and hence

$$P(A)-P(A \cap B) \le 1-P(B)$$

i.e., $P(A)+P(B)-1 \le P(A \cap B)$

or $\mathbf{P(A \cap B) \ge P(A)+P(B)-1}$

2. $$P(\overline{A}/B) = \frac{P(\overline{A} \cap B)}{P(B)} \qquad \ldots(2)$$

From the figure, $B = (A \cap B) \cup (\overline{A} \cap B)$

$\Rightarrow \quad P(B) = P(A \cap B)+P(\overline{A} \cap B)$

or $\quad P(\overline{A} \cap B) = P(B)-P(A \cap B)$

Hence (2) becomes, $P(\overline{A}/B) = \dfrac{P(B)-P(A \cap B)}{P(B)}$

Thus $\mathbf{P(\overline{A}/B) = 1-\dfrac{P(A \cap B)}{P(B)}}$

---

27. *If A, B, C are any three events, prove that*
$P(A \cup B/C) = P(A/C)+P(B/C)-P(A \cap B/C)$

>> $$P(A \cup B/C) = \frac{P[(A \cup B) \cap C]}{P(C)}$$

$$= \frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} \quad \text{by distributive law.}$$

$$= \frac{P(A \cap C)+P(B \cap C)-P[(A \cap C) \cap (B \cap C)]}{P(C)}$$

where we have used the addition rule.

Also $(A \cap C) \cap (B \cap C) = (A \cap B) \cap C$

$$\therefore \quad P(A \cup B/C) = \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P[(A \cap B) \cap C]}{P(C)}$$

Thus $\mathbf{P(A \cup B/C) = P(A/C)+P(B/C)-P(A \cap B/C)}$

---

28. *If A and B are events with* $P(A) = 3/8$, $P(B) = 5/8$ *and* $P(A \cup B) = 3/4$. *find* $P(A/B)$ *and* $P(B/A)$

>> $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ and $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

Consider $P(A \cup B) = P(A)+P(B)-P(A \cap B)$

That is, $3/4 = 3/8 + 5/8 - P(A \cap B)$

$\therefore \quad P(A \cap B) = 1/4$

Hence we have, $P(A/B) = \dfrac{1/4}{5/8}$ and $P(B/A) = \dfrac{1/4}{3/8}$

Thus $\mathbf{P(A/B) = 2/5}$ **and** $\mathbf{P(B/A) = 2/3}$

---

29. *Given,* $P(A) = 3/4$, $P(B) = 1/5$ *and* $P(A \cap B) = 1/20$, *find* $P(A \cup B)$, $P(A \cap \overline{B})$, $P(\overline{A} \cap B)$, $P(A/B)$, $P(B/A)$, $P(\overline{A}/\overline{B})$, $P(\overline{B}/\overline{A})$, $P(A/\overline{B})$ *and* $P(\overline{A}/B)$

>> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 3/4 + 1/5 - 1/20 = 9/10$

$\therefore \quad \mathbf{P(A \cup B) = 9/10}$

We have $P(A \cap \overline{B}) = P(A) - P(A \cap B)$ and

$P(\overline{A} \cap B) = P(B) - P(A \cap B)$

$P(A \cap \overline{B}) = 3/4 - 1/20$ and $P(\overline{A} \cap B) = 1/5 - 1/20$

$\therefore \quad \mathbf{P(A \cap \overline{B}) = 7/10 \; ; \; P(\overline{A} \cap B) = 3/20}$

Next, $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ and $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

$P(A/B) = \dfrac{1/20}{1/5}$ and $P(B/A) = \dfrac{1/20}{3/4}$

$\therefore \quad \mathbf{P(A/B) = 1/4 \; ; \; P(B/A) = 1/15}$

Next, $P(\overline{A}/\overline{B}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{B})}$ and $P(\overline{B}/\overline{A}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{A})}$

We know that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ *(De - Morgan's law)*

$\Rightarrow \quad P(\overline{A \cup B}) = P(\overline{A} \cap \overline{B})$

i.e., $1 - P(A \cup B) = P(\overline{A} \cap \overline{B})$

i.e., $1 - (9/10) = P(\overline{A} \cap \overline{B})$ or $P(\overline{A} \cap \overline{B}) = 1/10$

Hence $P(\overline{A}/\overline{B}) = \dfrac{1/10}{4/5} = \dfrac{1}{8}$, since $P(\overline{B}) = 1 - P(B) = 4/5$

Also $P(\overline{B}/\overline{A}) = \dfrac{1/10}{1/4} = \dfrac{2}{5}$, since $P(\overline{A}) = 1 - P(A) = = 1/4$

$\therefore \quad \mathbf{P(\overline{A}/\overline{B}) = 1/8 \; ; \; P(\overline{B}/\overline{A}) = 2/5}$

Next, $P(A/\overline{B}) = \dfrac{P(A \cap \overline{B})}{P(\overline{B})}$ and $P(\overline{A}/B) = \dfrac{P(\overline{A} \cap B)}{P(B)}$

$$P(A/\overline{B}) = \frac{7/10}{4/5} = \frac{7}{8} \quad \text{and} \quad P(\overline{A}/B) = \frac{3/20}{1/5} = \frac{3}{4}$$

$\therefore$ $\mathbf{P(A/\overline{B}) = 7/8\ ;\ P(\overline{A}/B) = 3/4}$

---

30. *Given that* $P(\overline{A} \cap \overline{B}) = 7/12,\ P(A \cap \overline{B}) = 1/6 = P(\overline{A} \cap B)$

1. *Prove that* A *and* B *are neither independent nor mutually exclusive.*
2. *Compute* $P(A/B) + P(B/A)$, $P(\overline{A}/\overline{B}) + P(\overline{B}/\overline{A})$ *and* $P(A/\overline{B}) + P(B/\overline{A})$

>> 1. $P(\overline{A \cup B}) = P(\overline{A} \cap \overline{B})$ by using De-Morgan's law.

i.e., $1 - P(A \cup B) = P(\overline{A} \cap \overline{B})$ or $P(A \cup B) = 1 - P(\overline{A} \cap \overline{B})$

$\therefore$ $P(A \cup B) = 1 - (7/12)$ or $P(A \cup B) = 5/12$

Let us consider $(A \cup \overline{A}) \cap \overline{B}$ and apply the distributive law.

i.e., $(A \cup \overline{A}) \cap \overline{B} = (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B})$ ...(1)

We note that $A \cup \overline{A} = S$ and $S \cap \overline{B} = \overline{B}$

Further we also note that $A \cap \overline{B}$ and $\overline{A} \cap \overline{B}$ are disjoint.

Hence (1) becomes

$$\overline{B} = (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B})$$

$\Rightarrow$ $P(\overline{B}) = P(A \cap \overline{B}) + P(\overline{A} \cap \overline{B})$

$= 1/6 + 7/12 = 9/12 = 3/4$

$\therefore$ $P(\overline{B}) = 3/4$ and hence $P(B) = 1/4$

Similarly $(B \cup \overline{B}) \cap \overline{A} = (B \cap \overline{A}) \cup (\overline{B} \cap \overline{A})$

LHS is $S \cap \overline{A} = \overline{A}$ and hence we have

$$P(\overline{A}) = P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B})$$

i.e., $P(\overline{A}) = 1/6 + 7/12 = 3/4$ and hence $P(A) = 1/4$

Consider $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

i.e., $5/12 = 1/4 + 1/4 - P(A \cap B)$

ie., $P(A \cap B) = 1/2 - 5/12$ or $P(A \cap B) = 1/12$

We know that,

A and B are independent if $P(A \cap B) = P(A) \cdot P(B)$

A and B are mutually exclusive if $P(A \cup B) = P(A) + P(B)$

We have $P(A \cap B) = 1/12$ and $P(A) \cdot P(B) = 1/16$

$P(A \cup B) = 5/12$ and $P(A) + P(B) = 1/2$

$P(A \cap B) \neq P(A) \cdot P(B)$ and $P(A \cup B) \neq P(A) + P(B)$

We conclude that A **and** B **are neither independent nor mutually exclusive.**

2. $$P(A/B) + P(B/A) = \frac{P(A \cap B)}{P(B)} + \frac{P(B \cap A)}{P(A)}$$
$$= \frac{1/12}{1/4} + \frac{1/12}{1/4} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Thus $\mathbf{P(A/B) + P(B/A) = 2/3}$

Next, $$P(\overline{A}/\overline{B}) + P(\overline{B}/\overline{A}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} + \frac{P(\overline{A} \cap \overline{B})}{P(\overline{A})}$$
$$= \frac{7/12}{3/4} + \frac{7/12}{3/4} = \frac{14}{9}$$

Thus $\mathbf{P(\overline{A}/\overline{B}) + P(\overline{B}/\overline{A}) = 14/9}$

Also, $$P(A/\overline{B}) + P(B/\overline{A}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} + \frac{P(\overline{A} \cap B)}{P(\overline{A})}$$
$$= \frac{1/6}{3/4} + \frac{1/6}{3/4} = \frac{4}{9}$$

Thus $\mathbf{P(A/\overline{B}) + P(B/\overline{A}) = 4/9}$

---

31. *If A, B are two events having* $P(A) = 1/2$, $P(B) = 1/3$ *and* $P(A \cap B) = 1/4$, *compute the following.*

(1) $P(A/B)$ **(2)** $P(B/A)$ **(3)** $P(\overline{A}/\overline{B})$ **(4)** $P(\overline{B}/\overline{A})$

\>> $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/3} = \mathbf{\frac{3}{4}}$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{1/2} = \mathbf{\frac{1}{2}}$$

$$P(\overline{A}/\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} \qquad \ldots(1)$$

But $P(\overline{B}) = 1 - P(B) = 1 - (1/3) = 2/3$

We have De-Morgan's law : $(\overline{A \cup B}) = \overline{A} \cap \overline{B}$

$\therefore \quad P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B).$

$$= 1 - \left[P(A) + P(B) - P(A \cap B) \right]$$

i.e., $P(\overline{A} \cap \overline{B}) = 1 - \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right] = \frac{5}{12}$

Substituting appropriate values in (1) we get

$$P(\overline{A}/\overline{B}) = \frac{5/12}{2/3} = \frac{5}{8}$$

Thus $\mathbf{P(\overline{A}/\overline{B}) = 5/8}$

Also $P(\overline{B}/\overline{A}) = \frac{P(\overline{B} \cap \overline{A})}{P(\overline{A})} = \frac{5/12}{1/2}$ since $P(\overline{A}) = 1 - P(A) = 1/2$

Thus $\mathbf{P(\overline{B}/\overline{A}) = 5/6}$

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32. *In a school 25% of the students failed in first language, 15% of the students failed in second language and 10% of the students failed in both. If a student is selected at random find the probability that*

(i) *He failed in first language if he had failed in the second language.*

(ii) *He failed in second language if he had failed in the first language.*

(iii) *He failed in either of the two languages.*

>> Let L_1 be the set of students failing in the first language and L_2 be the set of students failing in the second language. We have by data

$$P(L_1) = \frac{25}{100} = \frac{1}{4}, \quad P(L_2) = \frac{15}{100} = \frac{3}{20}, \quad P(L_1 \cap L_2) = \frac{10}{100} = \frac{1}{10}$$

(i) $P(L_1/L_2) = \frac{P(L_1 \cap L_2)}{P(L_2)} = \frac{1/10}{3/20} = \frac{2}{3}$

(ii) $P(L_2/L_1) = \frac{P(L_2 \cap L_1)}{P(L_1)} = \frac{1/10}{1/4} = \frac{2}{5}$

(iii) $P(L_1 \cup L_2) = P(L_1) + P(L_2) - P(L_1 \cap L_2)$

$\therefore$ $P(L_1 \cup L_2) = \frac{1}{4} + \frac{3}{20} - \frac{1}{10} = \frac{3}{10}$

---

33. *A solar water heater company manufactures two parts namely the heating pannel and the insulated tank. In the manufacturing process 9% are likely to be defective in the pannel and 5% are likely to be defective in the insulated tank. If an assembled unit is installed in a house what is the probability that it is non defective.*

>> Let $P(H)$ and $P(I)$ respectively denote the probability of heating pannel and insulated tank being defective.

$$P(H) = \frac{9}{100} = 0.09, \; P(I) = \frac{5}{100} = 0.05$$

We need to find $P(\overline{H} \cap \overline{I})$ as $\overline{H}$ and $\overline{I}$ respectively stands for non defective heating pannel and insulated tank.

We have by De-Morgan's law of sets, $\overline{A} \cap \overline{B} = (\overline{A \cup B})$

$\therefore \quad P(\overline{H} \cap \overline{I}) = P(\overline{H \cup I}) = 1 - P(H \cup I)$

ie., $\quad P(\overline{H} \cap \overline{I}) = 1 - \{P(H) + P(I) - P(H \cap I)\}$

$\quad P(\overline{H} \cap \overline{I}) = 1 - \{P(H) + P(I) - P(H) \cdot P(I)\}$

Since H and I are independent $P(H \cap I) = P(H) \cdot P(I)$

$\therefore \quad P(\overline{H} \cap \overline{I}) = 1 - \{0.09 + 0.05 - (0.09)(0.05)\} = \mathbf{0.8645}$

Aliter : $P(\overline{H} \cap \overline{I}) = P(\overline{H}) \cdot P(\overline{I})$ since $\overline{H}, \overline{I}$ are independent.

$\quad P(\overline{H} \cap \overline{I}) = [1 - P(H)][1 - P(I)]$

$\therefore \quad P(\overline{H} \cap \overline{I}) = (0.91)(0.95) = \mathbf{0.8645}$

---

34. *Three machines A, B and C produce respectively 60%, 30%, 10% of the total number of items of a factory. The percentages of defective output of these machines are respectively 2%, 3% and 4%. An item is selected at random and is found defective. Find the probability that the item was produced by machine C.*

>> Let A, B, C stand for the events of selection of an item from machines A, B, C

$$\therefore \quad P(A) = \frac{60}{100} = 0.6, \; P(B) = \frac{30}{100} = 0.3, \; P(C) = \frac{10}{100} = 0.1$$

Suppose D is the event of selection of a defective item then

$$P(D/A) = \frac{2}{100} = 0.02, \; P(D/B) = \frac{3}{100} = 0.03, \; P(D/C) = \frac{4}{100} = 0.04$$

To find the probability that a selected item is produced from the machine C, we need to find $P(C/D)$.

We have by Baye's theorem,

$$P(C/D) = \frac{P(C) \cdot P(D/C)}{P(A) \cdot P(D/A) + P(B) \cdot P(D/B) + P(C) \cdot P(D/C)}$$

$$= \frac{(0.1)(0.04)}{(0.6)(0.02) + (0.3)(0.03) + (0.1)(0.04)}$$

Thus $\quad P(C/D) = \mathbf{0.16}$

---

35. *In a bolt factory there are four machines A, B, C, D manufacturing respectively 20%, 15%, 25%, 40% of the total production. Out of these 5%, 4%, 3%, 2% are defective. If a bolt drawn at random was found defective what is the probability that it was manufactured by A or D ?*

>> By data A, B, C, D manufacture 20%, 15% 25% and 40% of the total production. Hence we have

$$P(A) = 0.2,\ P(B) = 0.15,\ P(C) = 0.25,\quad P(D) = 0.4$$

Let X be the event of selection of a defective bolt. Then

$$P(X/A) = 0.05,\ P(X/B) = 0.04,\ P(X/C) = 0.03,\ P(X/D) = 0.02$$

We need to compute $P(A \cup D/X)$

Since A and D are mutually exclusive we have,

$$P(A \cup D/X) = P(A/X) + P(D/X) \quad \ldots(1)$$

We have by Baye's theorem,

$$P(A/X) = \frac{P(A)\cdot P(X/A)}{P(A)\cdot P(X/A) + P(B)\cdot P(X/B) + P(C)\cdot P(X/C) + P(D)\cdot P(X/D)}$$

$$\therefore \quad P(A/X) = \frac{(0.2)(0.05)}{(0.2)(0.05) + (0.15)(0.04) + (0.25)(0.03) + (0.4)(0.02)}$$

i.e., $$P(A/X) = \frac{0.01}{0.0315} \approx 0.3175 \quad \ldots(2)$$

Also $$P(D/X) = \frac{P(D)P(X/D)}{0.0315} = \frac{(0.4)(0.02)}{0.0315} \approx 0.254 \quad \ldots(3)$$

We shall use (2) and (3) in (1).

Thus, $P(A \cup D/X) = 0.3175 + 0.254 = \mathbf{0.5715}$

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36. *An office has 4 secretaries handling respectively 20%, 60%, 15% and 5% of the files of all government reports. The probability that they misfile such reports are respectively 0.05, 0.1, 0.1 and 0.05. Find the probability that the misfiled report can be blamed on the first secretary.*

>> Let A_1, A_2, A_3, A_4 be the 4 secretaries of the office, respectively handling 20%, 60%, 15%, 5% of the files. Hence we have

$$P(A_1) = 20/100 = 0.2,\ P(A_2) = 0.6,\ P(A_3) = 0.15,\ P(A_4) = 0.05$$

Let E be the event of misfiling a report by the secretaries.

$$\therefore \quad P(E/A_1) = 0.05,\ P(E/A_2) = 0.1,\ P(E/A_3) = 0.1,\ P(E/A_4) = 0.05$$

We need to find $P(A_1/E)$ and we have by Baye's theorem,

$$P(A_1/E) = \frac{P(A_1)P(E/A_1)}{P(A_1)P(E/A_1)+P(A_2)P(E/A_2)+P(A_3)P(E/A_3)+P(A_4)P(E/A_4)}$$

$$= \frac{(0.2)(0.05)}{(0.2)(0.05)+(0.6)(0.1)+(0.15)(0.1)+(0.05)(0.05)}$$

Thus $\mathbf{P(A_1/E) = 0.1143}$

---

37. *The chance that a doctor will diagnose a disease correctly is 60%. The chance that a patient will die after correct diagnose is 40% and the chance of death by wrong diagnosis is 70%. If a patient dies, what is the chance that his disease was correctly diagnosed ?*

>> Let A be the event of correct diagnosis and B be the event of wrong diagnosis by the doctor.

$\therefore \quad P(A) = 0.6$ and $P(B) = 0.4$

Let E be the event that the patient dies.

$\therefore \quad P(E/A) = 0.4$ and $P(E/B) = 0.7$

We have to find $P(A/E)$ and by Baye's theorem,

$$P(A/E) = \frac{P(A)P(E/A)}{P(A)\cdot P(E/A)+P(B)\cdot P(E/B)}$$

$$= \frac{(0.6)(0.4)}{(0.6)(0.4)+(0.4)(0.7)} = 0.4615$$

Thus $\mathbf{P(A/E) = 0.4615}$

---

38. *Three machines A, B, C produces 50%, 30%, and 20% of the items in a factory. The percentage of defective outputs are 3, 4, 5. If an item is selected at random. What is the probability that it is defective ? What is the prabability that it is from A ?*

>> Machine A produce 50% of the items of the factory and out of these 3% are defective. Let D denote the event of selecting a defective item.

$\therefore \quad P(A) = 0.5$ and $P(D/A) = 0.03$

$P(B) = 0.3$ and $P(D/B) = 0.04$

$P(C) = 0.2$ and $P(D/C) = 0.05$

Now $\quad P(D) = P(A)P(D/A)+P(B)P(D/B)+P(C)P(D/C)$

$P(D) = (0.5)(0.03)+(0.3)(0.04)+(0.2)(0.05) = 0.037$

Thus the probability of selecting a defective item is 0.037

Next, we shall find the probability that the defective item is from A.
That is to find $P(A/D)$ and we have by Baye's theorem,

$$P(A/D) = \frac{P(A)P(D/A)}{P(D)} = \frac{(0.5)(0.03)}{0.037} = 0.4054$$

Thus $\mathbf{P(A/D) = 0.4054}$

---

39. *A bag contains three coins, one of which is two headed and the other two are normal and fair. A coin is chosen at random from the bag and tossed four times in succession. If head turns up each time, what is the probability that this is the two headed coin.*

>> Let C_1 be the two headed coin and C_2, C_3 be the normal coins.

Let E be the event of getting 4 heads in succession and we have to find $P(C_1/E)$

Now $P(E) = P(C_1)P(E/C_1) + P(C_2)P(E/C_2) + P(C_3)P(E/C_3)$

We have $P(C_1) = P(C_2) = P(C_3) = 1/3$

$P(E/C_1) = 1$ since C_1 is a two headed coin.

$P(E/C_2) = P(E/C_3) = 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 = 1/16$

since the probability of getting a head in a normal coin is 1/2

Hence $P(E) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{16} + \frac{1}{3} \cdot \frac{1}{16} = \frac{3}{8}$

Now $P(C_1/E) = \frac{P(C_1) \cdot P(E/C_1)}{P(E)} = \frac{1/3 \cdot 1}{3/8} = \frac{8}{9}$

Thus the required probability is 8/9

---

40. *Three major parties A, B, C are contending for power in the elections of a state and the chance of their winning the election is in the ratio 1 : 3 : 5. The parties A, B, C respectively have probabilities of banning the online lottery 2/3, 1/3, 3/5 What is the probability that there will be a ban on the online lottery in the state ? What is the probability that the ban is from the party C ?*

>> $P(A) = 1/9$, $P(B) = 3/9 = 1/3$, $P(C) = 5/9$

Let E be the event of banning the online lottery.

$\therefore$ $P(E/A) = 2/3$, $P(E/B) = 1/3$, $P(E/C) = 3/5$

Hence $P(E) = P(A)P(E/A) + P(B)P(E/B) + P(C)P(E/C)$

$$P(E) = \frac{1}{9} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{5}{9} \cdot \frac{3}{5} = \frac{14}{27}$$

Thus the probability of banning the online lottery is 14/27

Next we shall find $P(C/E)$ and we have by Baye's theorem,

$$P(C/E) = \frac{P(C)\,P(E/C)}{P(E)} = \frac{5/9 \cdot 3/5}{14/27} = \frac{9}{14}$$

Thus the probability o' ban from C is 9/14

---

41. *A company manufacturing ball pens in two writing colours blue and red make packets of 10 pens with 5 pens of each colour. In a particular shop it was found that after sales, packet 1 contained 3 blue and 2 red pens, packet 2 contained 2 blue and 3 red pens, packet 3 contained 3 blue and 5 red pens. On the demand of a customer for a pen, a packet was drawn at random and a pen was taken out. It was found blue. Find the probability that packet 1 was selected.*

\>> Let E_1, E_2, E_3 be the event of selecting packets 1, 2, 3 respectively at random.

$\therefore \quad P(E_1) = 1/3,\ P(E_2) = 1/3,\ P(E_3) = 1/3$

Let B be the event of selecting a blue pen.
Probability of selecting a blue pen from packet 1 is $P(B/E_1)$

$\therefore \quad P(B/E_1) = 3/5$. Similarly $P(B/E_2) = 2/5,\ P(B/E_3) = 3/8$

We have to find $P(E_1/B)$ and by Baye's theorem,

$$P(E_1/B) = \frac{P(E_1)\,P(B/E_1)}{P(E_1)\,P(B/E_1) + P(E_2)\,P(B/E_2) + P(E_3)\,P(B/E_3)}$$

$$= \frac{1/3 \cdot 3/5}{1/3 \cdot 3/5 + 1/3 \cdot 2/5 + 1/3 \cdot 3/8} = \frac{24}{55}$$

Thus $\mathbf{P(E_1/B) = 24/55}$

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42. *Ball pen refils are packed in pouches containing 25 refils in each pouch. In a shop it was found that 5 refils failed to write in pouch 1, 10 each in pouches 2 and 3, 1 refil in pouch 4. Suppose a refil is selected at random from one of the four pouches what is the probability that it fails to write ? What is the probability that it was from pouch 4?*

\>> Let E_1, E_2, E_3, E_4 be the event of selecting pouches at random.

$\therefore \quad P(E_1) = 1/4 = P(E_2) = P(E_3) = P(E_4)$

Let F be the event of selecting a refil failing to write

$\therefore \quad P(F/E_1) = 5/25,\ P(F/E_2) = 10/25,\ P(F/E_3) = 10/25,\ P(F/E_4) = 1/25$

Hence the probability of the event F is given by

$$P(F) = P(E_1)P(F/E_1) + P(E_2)P(F/E_2) + P(E_3)P(F/E_3) + P(E_4)P(F/E_4)$$

$$= \frac{1}{4}\left[\frac{5}{25} + \frac{10}{25} + \frac{10}{25} + \frac{1}{25}\right] = \frac{26}{100} = 0.26$$

Thus **$P(F) = 0.26$**

Next we have to find $P(E_4/F)$ and by Baye's theorem,

$$P(E_4/F) = \frac{P(E_4)P(F/E_4)}{P(F)} = \frac{1/4 \cdot 1/25}{26/100} = \frac{1}{26}$$

Thus **$P(E_4/F) = 0.03846 \approx 0.04$**

---

43. *In a class 70% are boys and 30% are girls. 5% of boys, 3% of the girls are irregular to the classes. What is the probability of a student selected at random is irregular to the classes and what is the probability that the irregular student is a girl ?*

Probability of selecting a boy $= P(B) = 70/100 = 0.7$

Probability of selecting a girl $= P(G) = 30/100 = 0.3$

Let I be the event of selecting an irregular student

$\therefore \quad P(I/B) = 5/100 = 0.05, \; P(I/G) = 3/100 = 0.03$

Hence $P(I) = P(B)P(I/B) + P(G)P(I/G)$

$$P(I) = (0.7)(0.05) + (0.3)(0.03) = 0.044$$

Thus the probability of selecting an irregular student is 0.044

Next we have to find $P(G/I)$ and by Baye's theorem,

$$P(G/I) = \frac{P(G)P(I/G)}{P(I)} = \frac{(0.3)(0.03)}{0.044} = 0.2045$$

Thus **$P(G/I) = 0.2045$**

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44. *In a college where boys and girls are equal in proportion, it was found that 10 out of 100 boys and 25 out of 100 girls were using the same brand of a two wheeler. If a student using that was selected at random what is the probability of being a boy ?*

>> $P(\text{Boy}) = P(B) = 1/2 = P(\text{Girl}) = P(G)$

Let E be the event of choosing a student using that brand of vehicle.

$\therefore \quad P(E/B) = 10/100 = 0.1$ and $P(E/G) = 25/100 = 0.25$

Now $\quad P(E) = P(B)P(E/B) + P(G)P(E/G)$

i.e., $\quad P(E) = 0.5\,[0.1 + 0.25] = 0.175$

We have to find $P(B/E)$ and by Baye's theorem,

$$P(B/E) = \frac{P(B)P(E/B)}{P(E)} = \frac{0.5 \times 0.1}{0.175} = 0.2857$$

Thus **$P(B/E) = 0.2857$**

Miscellaneous Problems

45. *The probabilities of n independent events are $p_1, p_2, p_3 \cdots p_n$. Find the probability of the happening of atleast one of the events.*

>> Let $E_1, E_2, \cdots E_n$ be the n independent events and we have by data,

$$P(E_1) = p_1,\ P(E_2) = p_2,\ \cdots P(E_n) = p_n$$

$$\Rightarrow \quad P(\overline{E}_1) = (1-p_1),\ P(\overline{E}_2) = (1-P_2),\ \cdots P(\overline{E}_n) = (1-p_n)$$

Probability of the happening of atleast one of the events $P(E)$

= 1 − (Probability of the non happening of any of the event)

$$P(E) = 1 - P(\overline{E_1 \cup E_2 \cdots \cup E_n})$$

$$= 1 - P(\overline{E}_1 \cap \overline{E}_2 \cap \cdots \overline{E}_n)$$

$$= 1 - P(\overline{E}_1) \cdot P(\overline{E}_2) \cdots P(\overline{E}_n)$$

Thus $\mathbf{P(E) = 1 - \{(1-p_1)(1-p_2)\cdots(1-p_n)\}}$

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46. *The probability that 3 students A, B, C solve a problem are 1/2, 1/3, 1/4 respectively. If the problem is simultaneously assigned to all of them, what is the probability that the problem is solved ?*

>> In general if E be the event of solving the problem and $\overline{E}$ is the event of not solving the problem, we have by data,

$P(A) = 1/2,\ P(B) = 1/3,\ P(C) = 1/4$ and hence

$P(\overline{A}) = 1/2,\ P(\overline{B}) = 2/3,\ P(\overline{C}) = 3/4$

***Method* - 1**

$$P(E) = P(A)P(B)P(C) + P(A)P(B)P(\overline{C}) + P(B)P(C)P(\overline{A})$$

$$+ P(C)P(A)P(\overline{B}) + P(\overline{A})P(\overline{B})P(C) + P(\overline{B})P(\overline{C})P(A) + P(\overline{C})P(\overline{A})P(B)$$

$$\therefore \quad P(E) = \frac{1}{24} + \frac{1}{8} + \frac{1}{24} + \frac{1}{12} + \frac{1}{12} + \frac{1}{4} + \frac{1}{8} = \frac{18}{24} = \frac{3}{4}$$

Thus $P(E)$ = **Probability of the problem being solved is 3/4**

Method - 2 This method is easier as we use the concept of Problem - 45.

$P(E)+P(\bar{E}) = 1$

$P(\bar{E})$ is the probability that the problem is not solved.

$\therefore \quad P(\bar{E}) = P(\bar{A})\cdot P(\bar{B})\cdot P(\bar{C}) = 1/2\cdot 2/3\cdot 3/4 = 1/4$

Hence $P(E) = 1-(1/4) \quad \therefore \quad \mathbf{P(E) = 3/4}$

Remark : *We have earlier solved the same type of problem [Ref. Problem - 8]*

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47. *A shooter can hit a target in 3 out of 4 shots and another shooter can hit the target in 2 out of 3 shots. Find the probability that the target is being hit* ***(a)*** *when both of them try* ***(b)*** *by only one shooter.*

>> Let S_1 and S_2 be the events that the shooters 1 and 2 hit the target.

$\therefore \quad P(S_1) = 3/4$ and $P(S_2) = 2/3$

(a) $P(S_1 \cup S_2) = P(S_1)+P(S_2)-P(S_1 \cap S_2)$.

But S_1, S_2 are independent.

$$\therefore \quad P(S_1 \cup S_2) = P(S_1)+P(S_2)-P(S_1)\cdot P(S_2)$$

$$= \frac{3}{4}+\frac{2}{3}-\frac{3}{4}\cdot\frac{2}{3} = \frac{11}{12}$$

Thus $\mathbf{P(S_1 \cup S_2) = 11/12}$

(b) Target being hit by only one shooter means

$(S_1 \cap \bar{S}_2) \cup (\bar{S}_1 \cap S_2)$

$$P[(S_1 \cap \bar{S}_2) \cup (\bar{S}_1 \cap S_2)] = P(S_1 \cap \bar{S}_2)+P(\bar{S}_1 \cap S_2)]$$

$$= P(S_1)\cdot P(\bar{S}_2)+P(\bar{S}_1)\cdot P(S_2)$$

$$= \frac{3}{4}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{2}{3} = \frac{5}{12}$$

Thus **the required probability is 5/12**

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48. *The probability that a team wins a match is 3/5. If this team play 3 matches in a tournament, what is the probability that the team*

(a) *win all the matches* *(b)* *win atleast one match*
(c) *win atmost one match* *(d)* *lose all the matches*

>> Let W be the event of winning a match by the team.

$P(W_1) = P(W_2) = P(W_3) = 3/5$

$\therefore \quad P(\overline{W}_1) = P(\overline{W}_2) = P(\overline{W}_3) = 2/5$

(a) Probability of winning all the matches

$$= P(W_1) \cdot P(W_2) \cdot P(W_3) = \mathbf{27/125}$$

(b) Probability of winning atleast one match

$$= 1 - \text{Probability of losing all the matches.}$$

$$= 1 - P(\overline{W}_1) \cdot P(\overline{W}_2) \cdot P(\overline{W}_3)$$

$$= 1 - (8/25) = \mathbf{17/25}$$

(c) Probability of winning atmost one match.

$$= P(\overline{W}_1) P(\overline{W}_2) P(\overline{W}_3) + P(W_1) P(\overline{W}_2) P(\overline{W}_3) + P(\overline{W}_1) P(W_2) P(\overline{W}_3) + P(\overline{W}_1) P(\overline{W}_2) P(W_3)$$

$$= \frac{8}{125} + 3\left[\frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}\right] = \frac{8}{125} + \frac{36}{125} = \mathbf{\frac{44}{125}}$$

(d) Probability of losing all the matches

$$= P(\overline{W}_1) P(\overline{W}_2) P(\overline{W}_3) = \mathbf{8/125}$$

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49. *The odds that a book will be reviewed favourably by 3 independent critics are 5 to 2, 4 to 3 and 3 to 4. Find the probability that majority of the reviews will be favourable.*

>> Let E_1, E_2, E_3 be the events of favourable review by the three critics respectively.

$\therefore \quad P(E_1) = 5/7,\ P(E_2) = 4/7,\ P(E_3) = 3/7$

$\Rightarrow \quad P(\overline{E}_1) = 2/7,\ P(\overline{E}_2) = 3/7,\ P(\overline{E}_3) = 4/7$

Majority of the reviews are favourable means that atleast two of three reviews should be favourable and if E denotes this event then we have

$$P(E) = P(E_1) P(E_2) P(\overline{E}_3) + P(E_2) P(E_3) P(\overline{E}_1) + P(E_3) P(E_1) P(\overline{E}_2) + P(E_1) \cdot P(E_2) \cdot P(E_3)$$

$$= \frac{5}{7} \cdot \frac{4}{7} \cdot \frac{4}{7} + \frac{4}{7} \cdot \frac{3}{7} \cdot \frac{2}{7} + \frac{3}{7} \cdot \frac{5}{7} \cdot \frac{3}{7} + \frac{5}{7} \cdot \frac{4}{7} \cdot \frac{3}{7} = \frac{209}{343}$$

Thus the required probability $P(E) = \mathbf{209/343 \approx 0.61}$

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50. *Three students A, B, C write an entrance examination. Their chances of passing are 1/2, 1/3 and 1/4 respectively. Find the probability that* ***(a)*** *atleast one of them passes* ***(b)*** *all of them passes* ***(c)*** *atleast two of them passes.*

>> Let E be the event of passing the examination by a student.

$\therefore \quad P(A) = 1/2\ ,\ P(B) = 1/3\ ,\ P(C) = 1/4$

$\Rightarrow \quad P(\overline{A}) = 1/2\ ,\ P(\overline{B}) = 2/3\ ,\ P(\overline{C}) = 3/4$

(a) Probability of atleast one of them passing.

$= 1 - (\text{Probability of none of them passing})$

$= 1 - P(\overline{A}) \cdot P(\overline{B}) \cdot P(\overline{C})$

$= 1 - (1/2 \cdot 2/3 \cdot 3/4) = 1 - (1/4) = \mathbf{3/4}$

(b) Probability of all of them passing

$= P(A) \cdot P(B) \cdot P(C)$

$= 1/2 \cdot 1/3 \cdot 1/4 = \mathbf{1/24}$

(c) Probability of atleast two of them passing

$$= P(A)P(B)P(\overline{C}) + P(B)P(C)P(\overline{A}) + P(C)P(A)P(\overline{B}) + P(A) \cdot P(B) \cdot P(C)$$

$= 3/24 + 1/24 + 2/24 + 1/24 = \mathbf{7/24}$

EXCERCISES

1. For any two events A and B, show that

   $P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$

2. If A and B are any two events with $B \subset A$ show that $P(A \cap \overline{B}) = P(A) - P(B)$ and hence deduce that $P(B) \le P(A)$

3. If $P(A) = P(B) = P(A \cap B)$ show the following result. $P(\overline{A} \cap B) + P(A \cap \overline{B}) = 0$

4. If A and B are events with $P(A) = 1/2$, $P(B) = 2/3$ and $P(A \cap B) = 1/4$, find $P(A \cup B)$, $P(B/A)$, $P(A/B)$ and $P(\overline{A}/\overline{B})$

5. If $P(A \cup B) = 5/6$, $P(A \cap B) = 1/3$ and $P(\overline{B}) = 1/2$ Compute $P(A)$ and $P(B)$. Hence show that A and B are independent.

6. Urn 1 contains 2 white and 3 black balls. Urn 2 has 4 white and 1 black ball and urn 3 has 3 white and 4 black balls. An urn is selected at random and a ball drawn from it was found white. Find the probability that urn 1 was selected.

7. A machine M_1 produces 1000 articles of which 20 are defective, M_2 produces 4000 articles of which 40 are defective, M_3 produces 5000 articles of which 50 are

defective. All these articles are piled up and an article picked from this pile was found defective. What is the probability that it is from M_1 ?

8. In a bolt factory 25%, 35% and 40% of the total is manufactured by machines A, B, C respectively out of which 5%, 4%, 2% are defective. What is the probability that a bolt drawn at random is defective ? What is the probability that it is from A ?

9. Boxes B_1, B_2, B_3 contain white (W), black (B) and red (R) balls as follows: B_1 : 2 W, 1 B, 2 R, B_2 : 3 W, 2 B, 4 R, B_3 : 4 W, 3 B, 2 R.
A 'die' is rolled and B_1 is selected if the number is 1 or 2, B_2 if the number is 3 or 4, B_3 if the number is 5 or 6. If a ball is drawn from the box thus selected, find the probability that the ball is from the box B_2 being a black ball.

10. 5 men out of 100 and 25 women out of 100 wear spectacles. If a person wearing spectacles is chosen at random, what is the probability of his being a man assuming that men and women are in equal proportion.

11. The chances that 4 students A, B, C, D solve a problem are 1/2, 1/3, 1/4, 1/4 respectively. If all of them try to solve the problem what is the probability that the problem is solved ?

12. A, B, C are three horses in a derby race. The probability of A winning is twice that of B and B winning is twice that of C. What are the probabilities of A, B, C to win the race ? What is the probability of A to lose the race ?

13. There are 4 coins of which one is a false coin with head on both sides. A coin is chosen at random and tossed 4 times. If head occurs all the 4 times what is the probability that the false coin has been chosen ?

14. There are 10 students of which 3 are graduates. If a committee of five students is to be formed what is the probability that there are (a) only 2 graduates (b) atleast 2 graduates.

15. The probability that an error is pointed out in the accounting statement prepared by A is 0.2 and that of B and C are 0.25 and 0.4 respectively. A, B, C respectively prepared 10, 16, 20 accounting statements. Find the expected number of correct statements in all.

ANSWERS

4. 11/12, 1/2, 3/8, 1/4
5. 2/3, 1/2
6. 14/57
7. 2/11
8. 25/69
9. 5/17
10. 4/25
11. 13/16
12. 4/7, 2/7, 1/7 ; 3/7
13. 16/19
14. 5/12, 1/12
15. 32